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Counting BPS states on the Enriques Calabi-Yau

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Abstract

We study topological string amplitudes for the FHSV model using various techniques. This model has a type II realization involving a Calabi-Yau threefold with Enriques fibres, which we call the Enriques Calabi-Yau. By applying heterotic/type IIA duality, we compute the topological amplitudes in the fibre to all genera. It turns out that there are two different ways to do the computation that lead to topological couplings with different BPS content. One of them gives the standard D0-D2 counting amplitudes, and from the other one we obtain information about bound states of D0-D4-D2 branes on the Enriques fibre. We also study the model using mirror symmetry and the holomorphic anomaly equations. We verify in this way the heterotic results for the D0-D2 generating functional for low genera and find closed expressions for the topological amplitudes on the total space in terms of modular forms, and up to genus three. This model turns out to be much simpler than the generic B-model and might be exactly solvable.

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Contents

1 Introduction

The solution of topological string theory on Calabi-Yau (CY) manifolds is an important problem with applications both in string theory and in enumerative geometry. Impressive tools have been developed in an extensive effort over the last twelve years, most

notably mirror symmetry, localization, deformation, large N-dualities, cohomological calculations in D2-D0 brane moduli spaces, and heterotic/type II duality. Nevertheless, a complete solution for the topological string amplitudes on compact CY manifolds is presently out of reach[2]. Applicable to the compact case are B-model calculations based on mirror symmetry and the holomorphic anomaly equation of [8, 31], and A-model calculations based on deformation arguments and relative Gromov-Witten invariants, which are calculated by localization [20, 44]. Both methods calculate the amplitudes genus by genus. The former provides contributions in all degrees at once, but only up to a holomorphic function (the so-called holomorphic ambiguity), whose determination requires further finite amount of data, which in practice are provided in a rather unsystematic case by case analysis. The A-model calculation proceeds degree by degree and the combinatorial complexity is in general prohibitive. Many (potentially all) compact CY manifolds are connected by transitions through complex degenerations. The behaviour of the topological string amplitudes at these transitions is relatively well understood. In view of this situation it is important to identify *the* compact CY manifold where the topological string is most tractable.

There is a compact example where topological string theory is exactly solvable, namely K3$\times\mathbb{T}^2$. The topological string amplitudes are all zero for genus $g \geq 2$, at $g = 0$ one has just the classical piece of the prepotential, and for $g = 1$ one just has the elliptic η function typical of the two-torus [7]. Hence this example is too simple, and this is due to the extended $\mathcal{N} = 4$ supersymmetry of the corresponding type II theory, related in turn to the $SU(2)$ holonomy. $\mathcal{N} = 2$ supersymmetry and the generic $SU(3)$ holonomy can be obtained by fibering K3 over $\mathbb{P}^1$. In these examples one can use heterotic/type II duality or special properties of the Hilbert scheme of complex surfaces to write down explicitly all genus topological amplitudes for the classes in the K3 fiber [43, 33]. However the decisive step in going from the surface to the threefold, i.e. the inclusion of the base and mixed classes, is hard. Results up to $g = 2$ have been obtained in [33].

This motivates to consider the problem on a special CY with intermediate holonomy $SU(2) \times \mathbb{Z}_2$ constructed in [10, 53, 18], as an orbifold w.r.t. a free $\mathbb{Z}_2$ involution of K3$\times\mathbb{T}^2$. The resulting space exhibits a K3 fibration with four fibres of multiplicity two over the four fixed points of the involution in the base, which are Enriques surfaces. A good deal of the nontrivial geometry of this CY comes from the geometry of the Enriques fibers, and we will call it the *Enriques CY manifold*. The string vacuum obtained by compactifying type II theory on the Enriques CY has $\mathcal{N} = 2$ supersymmetry and is known as the FHSV model. The $\mathbb{Z}_2$ lifts instanton zero modes related to the

[2] In contrast, on non-compact toric Calabi-Yau manifolds the problem is completely solved by localization, and much more efficiently by using the topological vertex [1], which relates the genus expansion to a $1/N$ expansion of Chern-Simons theory.

$\mathbb{T}^2$ so that simple instanton effects which cancel in the $\mathcal{N}=4$ theory contribute to the $\mathcal{N}=2$ effective action. The Enriques CY seems to be the simplest CY compactification with nontrivial topological string amplitudes. Moreover it has a dual description as an asymmetric orbifold of the heterotic string [18]. Various aspects of the FHSV model have been studied in the past, see for example [4, 13]. In particular, the genus one topological string amplitude of the FHSV model was determined by Harvey and Moore in [26].

This paper makes a first step to determine the topological string amplitudes of the FHSV model using heterotic/type II duality, B-model techniques and the cohomology of D2-D0 brane moduli spaces. Although we haven't been able to solve the model in full, we will present various results which show that indeed the model has some simplifying features that might lead to a complete solution. The simplicity of the model is also apparent from a mathematical point of view, and it turns out that the techniques developed in [20, 44] lead to simple recursive formulae for the Gromov-Witten invariants of the Enriques CY at low genera [45].

A heterotic one-loop calculation is used to determine the F_g couplings in the fibre direction, using the techniques developed in [24, 3, 43]. It turns out that this calculation can be made in two different ways, which we call the *geometric reduction* and the *Borcherds-Harvey-Moore (BHM) reduction*. The resulting expressions are appropriate for different regions in moduli space, and they turn out to have a different enumerative meaning. The result obtained in the geometric reduction corresponds to the large radius limit, reproduces the geometric expectations one has for a generating functional of Gromov-Witten invariants, and as explained in [21] counts D0-D2 bound states. We suggest that the result obtained in the BHM reduction is related to a counting of D0-D2-D4 bound states in a different region of moduli space. The result of [26] for the genus one amplitude was in fact implicitly obtained in the BHM reduction.

We then study the model by using mirror symmetry and the holomorphic anomaly equations. To do that, we first find an algebraic realization of the CY manifold involved in the FHSV model, and we find its mirror by using standard techniques. This leads to a model which is still very difficult to solve due to the presence of ten deformation parameters. To avoid this problem, we find a reduced model with only two fibre parameters which is obtained by blowing down the E_8 part of the homology of the original type A model. This model turns out to be very tractable, and all relevant quantities can be expressed in closed form in terms of modular forms. We present explicit formulae for the topological string amplitudes in the fiber up to genus 3 which agree with the predictions of heterotic/type II duality. The holomorphic anomaly equations turn out to be extremely simple due to various exceptional properties of the model (like the absence of worldsheet instanton corrections for the prepotential, already pointed out in [18]). Although we do not have enough information to fix the

holomorphic ambiguity in the base (except at genus 2, where explicit results have been obtained in [45]), we make a natural conjecture that leads to consistent results in genus three and four and might hold in general.

The organization of this paper is as follows. In section 2 we review the heterotic computation of topological string amplitudes in K3$\times\mathbb{T}^2$ compactifications. In section 3 we present various results on the FHSV model, both in its type IIA and its heterotic incarnations. In section 4 we compute the topological string amplitudes in the heterotic theory, in both the geometric and the BHM reductions. In section 5 we present an interpretation of the results in the geometric reduction in terms of BPS invariants associated to D2-D0 bound states, following [21, 31]. In section 6 we study the mirror B-model for a "reduced" version of the theory, and we compare the results with those obtained in the heterotic computation. Appendix A collects some useful formulae for modular forms. Appendix B summarizes some results about the lattice reduction technique which is used in the heterotic computation.

2 Heterotic/type II duality and F_g couplings

2.1 The F_g couplings in heterotic string theory

The duality between heterotic compactifications on K3$\times\mathbb{T}^2$ and type II theory on Calabi-Yau's which are K3 fibrations [30] has been a source of very rich information in string theory (see for example [41] for a review of results). One of the most interesting applications of this duality is the computation of the topological string amplitudes F_g. These F_g couplings are F-terms for compactifications of type II theory on Calabi-Yau manifolds [8, 2], and they give terms in the four-dimensional effective action of the form

$$\int F_g(t,\bar{t})T^{2g-2}R^2 + \cdots \tag{2.1}$$

where T is the graviphoton field strength and R is the Riemann curvature. It turns out that, on the heterotic side, all these couplings appear at one-loop [3] and can be computed in closed form [43]. In this section we will briefly review the computation of the F_g amplitudes by using heterotic/type II duality. One drawback of this duality is that it only enables us to compute this amplitude in the limit of infinite volume for the basis of the K3 fibration. This is due to the fact that, under heterotic/type IIA duality, the heterotic dilaton S is identified with the complexified area of the base $\mathbb{P}^1$ of the fibration,

$$\mathrm{vol}_{\mathbb{C}}(\mathbb{P}^1) = 4\pi S = \frac{4\pi}{g_{\mathrm{het}}^2}. \tag{2.2}$$

Therefore the perturbative regime of the heterotic string corresponds to the limit $\mathrm{vol}_{\mathbb{C}}(\mathbb{P}^1) \to \infty$. On the other hand, the duality gives closed, elegant expressions for all the F_g amplitudes restricted to fiber classes in terms of modular forms. These classes correspond to the Picard lattice of the K3 fibre, which will be denoted by Pic(K3) (more precisely, one has to consider the monodromy-invariant part of the Picard lattice).

Before stating the main results, we introduce some notation on Narain lattices and Siegel-Narain theta functions. Given a lattice Γ of signature (b_+, b_-), a projection P is an orthogonal decomposition of $\Gamma \otimes \mathbb{R}$ into subspaces of definite signature:

$$P : \Gamma \otimes \mathbb{R} \simeq \mathbb{R}^{b_+} \perp \mathbb{R}^{b_-}.$$

We will denote by $p_\pm = P_\pm(p)$ the projections onto the two factors. The Siegel-Narain theta function is defined as

$$\Theta_\Gamma(\tau, \alpha, \beta) = \sum_{p \in \Gamma} \exp\left\{ \pi i \tau (p + \beta/2)_+^2 + \pi i \overline{\tau} (p + \beta/2)_-^2 + \pi i (p + \beta/2, \alpha) \right\}. \tag{2.3}$$

When $\alpha = \beta = 0$, we will simply write $\Theta_\Gamma(\tau)$. As usual, we write $q = \exp(2\pi i \tau)$, and $\tau_2 = \mathrm{Im}\, \tau$.

We will consider compactifications of the heterotic string on K3$\times\mathbb{T}^2$ and orbifolds thereof. These compactifications lead to effective theories with $\mathcal{N} = 2$ supersymmetry in four dimensions, and they involve Narain lattices with $b^+ = 2$, which can be identified with the two right-moving directions along $\mathbb{T}^2$. Therefore, we can identify $\mathbb{R}^{b_+} \simeq \mathbb{C}$, and we will represent $p_+ \in \mathbb{R}^2$ by $p_R \in \mathbb{C}$, so that p_+^2 is given by $|p_R|^2$.

The general expression for the F_g couplings in these compactifications is given by the one-loop integral [3] (see [32] for an excellent introduction to one-loop corrections in string theory)

$$F_g = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \frac{1}{|\eta|^4} \sum_{\text{even}} \frac{i}{\pi} (-1)^{a+b+ab} \partial_\tau \left(\frac{\vartheta[^a_b](\tau)}{\eta(\tau)} \right) Z_g^{\text{int}}[^a_b]. \tag{2.4}$$

In this equation, the integration is over the fundamental domain of the torus,

$$Z_g^{\text{int}}[^a_b] = \langle : \left(\partial X\right)^{2g} : \rangle \tag{2.5}$$

is a correlation function evaluated in the internal conformal field theory, and X is the complex boson corresponding to the right-moving modes on the $\mathbb{T}^2$. The evaluation of the correlation function reduces to zero modes [3], and the final result involves insertions of the right-moving momentum p_R. For this reason, it is convenient to introduce the Narain theta function with an insertion,

$$\Theta_\Gamma^g(\tau, \alpha, \beta) = \sum_{p \in \Gamma} p_R^{2g-2} \exp\left\{ \pi i \tau (p + \beta/2)_+^2 + \pi i \overline{\tau} (p + \beta/2)_-^2 + \pi i (p + \beta/2, \alpha) \right\}. \tag{2.6}$$

In general the internal CFT will be an orbifold theory and we will have to consider different orbifold blocks, which will be labelled by J. For each of these blocks there is a different Narain lattice Γ_J with different α_J, β_J, and we will denote

$$\Theta^g_J = \Theta^g_{\Gamma_J}(\tau, \alpha_J, \beta_J). \tag{2.7}$$

The integral (2.4) can now be written as [3, 48]

$$F_g = \int_{\mathcal{F}} d^2\tau \tau_2^{2g-3} \sum_J \mathcal{I}^g_J, \tag{2.8}$$

where

$$\mathcal{I}^g_J = \frac{\widehat{\mathcal{P}}_g(q)}{Y^{g-1}} \overline{\Theta}^g_J(\tau) f_J(q). \tag{2.9}$$

In this equation, $\widehat{\mathcal{P}}_g(q)$ is a one-loop correlation function of the bosonic fields and is given by [37, 3]

$$\mathrm{e}^{-\pi\lambda^2\tau_2} \left(\frac{2\pi\eta^3\lambda}{\vartheta_1(\lambda|\tau)} \right)^2 = \sum_{g=0}^{\infty} (2\pi\lambda)^{2g} \widehat{\mathcal{P}}_g(q). \tag{2.10}$$

$f_J(q)$ is a modular form which depends on the details of the internal CFT. Finally, the quantity Y in (2.9) is a moduli-dependent function related to the Kähler potential as $K = -\log Y$. We will also define the holomorphic counterpart $\mathcal{P}_g(q)$ of $\widehat{\mathcal{P}}_g(q)$ by

$$\left(\frac{2\pi\eta^3\lambda}{\vartheta_1(\lambda|\tau)} \right)^2 = \sum_{g=0}^{\infty} (2\pi\lambda)^{2g} \mathcal{P}_g(q). \tag{2.11}$$

The quantities $\mathcal{P}_g(q)$ can be explicitly written in terms of generalized Eisenstein series. To do this, one uses the expansion

$$\frac{2\pi\eta^3 z}{\vartheta_1(z|\tau)} = -\exp\left[\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k} E_{2k}(\tau) z^{2k} \right]. \tag{2.12}$$

If we now introduce the polynomials $\mathcal{S}_k$ through:

$$\exp\left[\sum_{n=1}^{\infty} x_n z^n \right] = \sum_{n=0}^{\infty} \mathcal{S}_n(x_1, \ldots, x_n) z^n, \tag{2.13}$$

we can easily check that $\mathcal{P}_g(q)$ is a quasimodular form of weight $(2g, 0)$ given by

$$\mathcal{P}_g(q) = \mathcal{S}_g\left(x_k = \frac{|B_{2k}|}{k(2k)!} E_{2k}(q) \right). \tag{2.14}$$

where B_{2k} are Bernoulli numbers, and $E_{2k}(q)$ is the Eisenstein series introduced in (A.9). We have, for instance,

$$\mathcal{P}_1(q) = \frac{1}{12} E_2(q), \quad \mathcal{P}_2(q) = \frac{1}{1440}(5E_2^2 + E_4). \tag{2.15}$$

The non-holomorphic modular forms $\widehat{\mathcal{P}}_g$ are obtained by an equation identical to (2.14) after changing E_2 by

$$\widehat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi\tau_2}. \tag{2.16}$$

The computation of the F_g amplitudes involves, then, the determination of the modular forms $f_J(q)$, and the evaluation of the integral over the fundamental domain. The first step is easy when there is an orbifold realization. The second step is more involved and requires the method first introduced in [17] in the context of string threshold corrections. This method was further refined and developed in [24, 12, 47]. We will refer to the approach presented in [24, 12, 47] to calculate these integrals as the *lattice reduction* technique. This technique, which was used to compute the F_g couplings in [43], computes the integral (2.8) iteratively by "integrating out" a sublattice of the Narain lattice of signature $(1,1)$, therefore reducing its rank at every step. In the cases considered in this paper, where one starts with lattices of signature $\Gamma^{2,2+s}$, it is sufficient to perform the lattice reduction once by "integrating out" a sublattice

$$\Gamma^{1,1} = \langle z, z' \rangle. \tag{2.17}$$

The generating vector z is called the reduction vector. The details of the lattice reduction procedure are rather intricate, and we present some of them in Appendix B. There are two general important properties of the result for F_g which are worth pointing out. The first one is that different choices for the sublattices $\Gamma^{1,1}$ to be integrated out in the process of lattice reduction lead in general to different results for the integral, and each of these expressions is valid in a different region of moduli space. The second property is that, although the result for the integral (2.8) is rather complicated, the holomorphic limit

$$\bar{t} \to \infty, \qquad t \text{ fixed}, \tag{2.18}$$

leads to a rather simple expression for F_g. This holomorphic limit is the one needed to extract the topological information of F_g [8].

We now present some general results on the holomorphic limit of F_g, obtained from a lattice reduction computation of the heterotic integral (2.8). For simplicity, we will restrict ourselves to the case in which one has a single lattice involved in the integrand (2.8), and $\alpha = \beta = 0$. In general, the integrand will be a sum over different orbifold blocks and different lattices Γ_J with nonvanishing α, β. The final answer for F_g in these cases will be given by a sum over the different blocks. The presence of α, β leads however to nontrivial modifications of the result, as it has been already noticed in various papers [48, 27, 47, 42, 40]. We will consider these modifications when we analyze the FHSV model. Most of the results we will present have been obtained in [43], although we will consider a slightly more general situation which will be needed for the FHSV model.

We first introduce some necessary ingredients to write down the answer. First of all, the norm $|z_+|^2$ of the projected reduction vector depends on the Narain moduli of the compactification as

$$|z_+|^2 = \frac{\nu}{Y}, \tag{2.19}$$

where $Y = \mathrm{e}^{-K}$ is the moduli-dependent quantity introduced in (2.9), and ν is a real number related to the norm of z. In the STU model considered in [43], $\nu = 1$, but as we will see in general it can take other values. We will label an element p^K of the reduced lattice by a vector r of integer coordinates. The resulting expression for the holomorphic limit of F_g depends on the moduli through the combination [43]

$$\exp\left(2\pi\mathrm{i}(p^K, \mu/N) + \frac{2\pi\widetilde{P}_+(p^K)}{|z_+|}\right). \tag{2.20}$$

The different ingredients in this expression are explained in detail in Appendix B. The first term of (2.20) comes from the exponent in the second line of (B.13), and the second term comes from the argument of the Bessel function in (B.13). One can easily seen, by using the explicit expressions for the different quantities involved, that the exponent in (2.20) can be written in the form

$$\nu^{-\frac{1}{2}} 2\pi\mathrm{i} r \cdot y, \tag{2.21}$$

where y is a vector of holomorphic coordinates for the heterotic moduli space (which is related to t, the flat coordinates in the positive Kähler cone in the type II realization, in a simple way). We will see concrete examples of this in the calculation for the FHSV model. Finally, we define the coefficients $c_g(n)$ through

$$\mathcal{P}_g(q) f(q) = \sum_n c_g(n) q^n. \tag{2.22}$$

The final expression for F_g is:

$$F_g(t) = \nu^{1-g} \sum_{r>0} c_g(r^2/2) \sum_{\ell=1}^{\infty} \ell^{2g-3} \mathrm{e}^{\ell \nu^{-\frac{1}{2}} 2\pi\mathrm{i} r \cdot y}. \tag{2.23}$$

In this equation, r^2 is computed with the norm of the reduced lattice K, and the restriction $r > 0$ means that we consider vectors such that $\mathrm{Im}(r \cdot y) > 0$, as well as a finite number of boundary cases [24, 43]. The sum over ℓ in (2.23) can be written as

$$\mathrm{Li}_{3-2g}(\mathrm{e}^{\nu^{-\frac{1}{2}} 2\pi\mathrm{i} r \cdot y}), \tag{2.24}$$

where Li_n is the polylogarithm of index n defined as

$$\mathrm{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}. \tag{2.25}$$

In the above expression for F_g we are not taking into account constant terms as well as polynomial terms in y and $\mathrm{Im}\, y$ which also appear in the heterotic computation [24, 43].

2.2 BPS content of the F_g couplings

As shown in [24, 25], the couplings F_g are BPS-saturated amplitudes and they can be regarded as generating functions that count in an appropriate way the BPS states of the $\mathcal{N} = 2$ compactification. The underlying structure of the couplings was further clarified in the work of Gopakumar and Vafa [21], who gave a precise formula for the BPS content of the F_g in terms of bound states of D0-D2 branes in a type IIA compactification on a CY threefold X. These bound states lead to BPS particles in four dimensions labelled by three quantum numbers. The first quantum number is the homology class $r \in H_2(X, \mathbb{Z})$ of the Riemann surface wrapped by the D2. The other two quantum numbers are given by the off-shell spin content j_L, j_R with respect to the algebra $\mathrm{su}(2)_L \times \mathrm{su}(2)_R$ of the rotation group $SO(4)$. Let us denote by $N_{j^3_L, j^3_R}(r)$ the number of BPS states with these quantum numbers. This number is not invariant under deformations, therefore [21] considered the index $n_g(r)$ defined by

$$\sum_{j^3_L, j^3_R} (-1)^{2j^3_R}(2j^3_R + 1) N_{j^3_R, J^3_L}(r)[\mathbf{j_L}] = \sum_{g=0}^{\infty} n_g(r) I_g \ , \tag{2.26}$$

where $I_g = \left[\left(\tfrac{\mathbf{1}}{\mathbf{2}}\right)_L + 2(\mathbf{0})_L\right]^{\otimes g}$. The integer numbers $n_g(r)$, which characterize the spectrum of D2-D0 bound states in CY compactifications of type IIA, are called *Gopakumar-Vafa (GV) invariants.*

Let us now consider the generating functional of topological string amplitudes

$$F(\lambda) = \sum_{g=0}^{\infty} F_g(t) \lambda^{2g-2}. \tag{2.27}$$

According to [21], the worldsheet instanton corrections to $F(\lambda)$ can be obtained by a Schwinger one-loop computation involving only the D2-D0 bound states:

$$F(\lambda) = \sum_{g=0}^{\infty} \sum_{r \in H_2(M,\mathbb{Z})} \sum_{m=-\infty}^{\infty} n_g(r) \int_0^{\infty} \frac{ds}{s} \left(2 \sin \frac{s}{2}\right)^{2g-2} \exp\left[-\frac{s}{\lambda}(r \cdot t + 2\pi \mathrm{i} m)\right]. \tag{2.28}$$

In this formula, the sum over m is over the number of D0 states bound to the D2s, and we have taken into account that the index $n_g(r)$ is independent of m. After a Poisson resummation over m one finds [21]:

$$F(\lambda) = \sum_{g=0}^{\infty} \sum_{r \in H_2(M,\mathbb{Z})} \sum_{d=1}^{\infty} n_g(r) \frac{1}{d} \left(2 \sin \frac{d\lambda}{2}\right)^{2g-2} \mathrm{e}^{-dr \cdot t}, \tag{2.29}$$

Notice that the sum over d in (2.29) plays the same role as the sum over ℓ in the heterotic computation (2.23). This expression, which takes into account the spectrum of "electric" states associated to D2-D0 branes, is valid in the large radius limit of the CY compactification, since in this region the lightest states are indeed the D2 and D0 branes and their bound states, while the D4 and D6 "magnetic" states are heavy. Equation (2.29) leads to strong structural predictions for the topological string amplitudes F_g when written in terms of GV invariants. Up to genus 4, one finds (for the instanton part)

$$\begin{aligned}
F_0 &= \sum_{r\in H_2(M,\mathbb{Z})} n_0(r)\mathrm{Li}_3(\mathrm{e}^{-r\cdot t}),\\
F_1 &= \sum_{r\in H_2(M,\mathbb{Z})} \left(\frac{n_0(r)}{12}+n_1(r)\right)\mathrm{Li}_1(\mathrm{e}^{-r\cdot t}),\\
F_2 &= \sum_{r\in H_2(M,\mathbb{Z})} \left(\frac{n_0(r)}{240}+n_2(r)\right)\mathrm{Li}_{-1}(\mathrm{e}^{-r\cdot t}),\\
F_3 &= \sum_{r\in H_2(M,\mathbb{Z})} \left(\frac{n_0(r)}{6048}-\frac{n_2(r)}{12}+n_3(r)\right)\mathrm{Li}_{-3}(\mathrm{e}^{-r\cdot t}),\\
F_4 &= \sum_{r\in H_2(M,\mathbb{Z})} \left(\frac{n_0(r)}{172800}+\frac{n_2(r)}{360}-\frac{n_3(r)}{6}+n_4(r)\right)\mathrm{Li}_{-5}(\mathrm{e}^{-r\cdot t}).
\end{aligned} \tag{2.30}$$

In the simple case where $\nu=1$, the heterotic result (2.23) leads to a simple generating function for the GV invariants. To see this, notice that, if we write

$$F(\lambda)=\sum_{g=0}^{\infty}\sum_{r\in H_2(M,\mathbb{Z})}\widehat{N}_g(r)\mathrm{Li}_{3-2g}(\mathrm{e}^{-r\cdot t})\lambda^{2g-2}, \tag{2.31}$$

then from (2.29) one has the following relation for fixed r

$$\sum_{g=0}^{\infty} n_g(r)\left(2\sin\frac{\lambda}{2}\right)^{2g-2}=\sum_{g=0}^{\infty}\widehat{N}_g(r)\lambda^{2g-2}, \tag{2.32}$$

Under heterotic/type II duality, and with an appropriate choice of lattice reduction, the reduced lattice K that appears in the heterotic computation becomes the Picard lattice of the K3 fiber Pic(K3), and the vectors r label homology classes in this lattice. According to (2.23), one has that $\widehat{N}_g(r)=c_g(r^2/2)$, where the $c_g(n)$ are defined in (2.22). If we now use (2.11) and the product representation of $\vartheta_1(\nu|\tau)$ given in (A.4), we find

$$\sum_{r\in\mathrm{Pic(K3)}}\sum_{g=0}^{\infty} n_g(r) z^g q^{r^2/2}=f(q)\xi^2(z), \tag{2.33}$$

where $z = 4\sin^2(\lambda/2)$, and $\xi(z)$ is the function that appears in helicity supertraces (see for example [32, 15])

$$\xi(z) = \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{1 - 2q^n \cos\lambda + q^{2n}} = \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-q^n)^2 + zq^n}. \tag{2.34}$$

A similar expression was written down in [29] in a particular example. Equation (2.33) applies to many different heterotic duals, like the ones studied in [43, 33]. Notice that, if the modular form $f(q)$ has an expansion in q with integer coefficients, then integrality of $n_g(r)$ is manifest. The final result for the generating functional of the $n_g(r)$ involves a model-dependent quantity (the modular form $f(q)$) as well as the universal factor $\xi^2(z)$. Therefore, in these heterotic models, the enumerative information of the F_gs is encoded in a single modular form $f(q)$, and this leads to a powerful principle which can be used to determine these couplings in a variety of models [33].

3 The FHSV model

In this section we will introduce and study the FHSV model of [18]. We will first discuss the heterotic side and give some details about the one-loop partition function which will be needed in the computation of the F_g couplings. Then we discuss the type IIA side and the geometry of the Calabi-Yau, which will be important to give an interpretation of the couplings and in the B-model analysis of section 6.

3.1 The heterotic side of the FHSV model

The FHSV model is defined, on the heterotic side, by an asymmetric orbifold [18]. One first considers the splitting of the compactification lattice $\Gamma^{6,22}$ as

$$\Gamma_u = \Gamma_1^{1,9} \oplus \Gamma_2^{1,9} \oplus \Gamma_s^{1,1} \oplus \Gamma^{2,2} \oplus \Gamma_g^{1,1}, \tag{3.1}$$

where each of the $\Gamma^{1,9}$ can be further decomposed as

$$\Gamma^{1,9} = \Gamma_d^{1,1} \oplus E_8(-1). \tag{3.2}$$

We now act with a $\mathbb{Z}_2$ symmetry as follows:

$$|p_1, p_2, p_3, p_4, p_5\rangle \to e^{\pi i \delta \cdot p_3} |p_2, p_1, p_3, -p_4, -p_5\rangle \tag{3.3}$$

where $\delta = (1, -1) \in \Gamma_s^{1,1}$, and $\delta^2 = -2$. Therefore $\mathbb{Z}_2$ acts as an exchange symmetry in the direct sum $\Gamma_1^{1,9} \oplus \Gamma_2^{1,9}$, as a shift in $\Gamma_s^{1,1}$, and as -1 in $\Gamma^{2,2} \oplus \Gamma_g^{1,1}$. It is easy to see [18] that this asymmetric orbifold leads to an heterotic string compactification

J	1	2	3
ζ_J	2	1/2	1/2
α_J	δ	0	δ
β_J	0	δ	δ

Table 1: ζ_J, α_J and β_J for the different blocks

with $\mathcal{N}=2$ supersymmetry in four dimensions. The massless spectrum consists of 11 vector multiplets, 11 hypermultiplets, and the supergravity multiplet.

The vector multiplet moduli space for this compactification is given by

$$\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/SO(2)\times\mathcal{M}, \tag{3.4}$$

where

$$\mathcal{M}=O(\Gamma_1)\backslash O(2,10)/[O(2)\times O(10)], \tag{3.5}$$

and $O(\Gamma_1)$ is the group of automorphisms of the lattice

$$\Gamma_1=\Gamma_s^{1,1}\oplus\Gamma_d^{1,1}(2)\oplus E_8(-2). \tag{3.6}$$

This is in fact the lattice associated to the untwisted, projected sector of the orbifold.

As a warm up exercise, we will now compute the one-loop partition function of the FHSV orbifold, since the results will be useful for the computation of the F_g amplitudes (the helicity supertrace generating function of this model has been independently computed in the recent paper [15]). We will denote by $Z[{h\atop g}]$ the partition functions on the sector twisted by h and with the g element inserted. Here, $g,h=0,1$ in the usual way.

Let us first consider the bosonic sector. In the untwisted, unprojected sector we simply have

$$Z^b[{0\atop 0}]=\frac{1}{2\bar\eta^{24}(\tau)\eta^8(\tau)}\overline{\Theta}_{\Gamma^u}(\tau). \tag{3.7}$$

In order to consider the other sectors, we introduce the lattices Γ_J with $J=1,2,3$:

$$\Gamma_J=\Gamma_s^{1,1}\oplus\Gamma_d^{1,1}(\zeta_J)\oplus E_8(-\zeta_J), \tag{3.8}$$

The values of ζ_J, α_J, β_J are given in table 1. The three different cases $J=1,2,3$ correspond respectively to the orbifold blocks 01, 10 and 11. In the untwisted, projected sector we identify the two sets of bosonic excitations associated to the two $\Gamma^{1,9}$ lattices. This amounts to a doubling of the τ parameter in the nonzero modes [36]. We then find,

$$Z^b[{0\atop 1}]=\frac{4}{\bar\eta^9(2\tau)\eta(2\tau)\bar\eta^3(\tau)\eta^3(\tau)}\left|\frac{\eta(\tau)}{\vartheta[{1\atop 0}](\tau)}\right|^3\overline{\Theta}_{\Gamma_1}(\tau,\delta,0). \tag{3.9}$$

For the 10 and 11 orbifold blocks we find

$$
\begin{aligned}
Z^b[{}^1_0] =& \frac{4}{\bar\eta^9(\tau/2)\eta(\tau/2)\bar\eta^3(\tau)\eta^3(\tau)}\left|\frac{\eta(\tau)}{\vartheta[{}^0_1](\tau)}\right|^3 \overline{\Theta}_{\Gamma_2}(\tau,0,\delta),\\
Z^b[{}^1_1] =& \frac{4}{\bar\eta^9(\frac{\tau+1}{2})\eta(\frac{\tau+1}{2})\bar\eta^3(\tau)\eta^3(\tau)}\left|\frac{\eta(\tau)}{\vartheta[{}^0_0](\tau)}\right|^3 \overline{\Theta}'_{\Gamma_3}(\tau,\delta,\delta).
\end{aligned}
\tag{3.10}
$$

In the 11 block, the $'$ in the theta function indicates that the sum over lattice vectors includes an insertion of

$$
(-1)^{v^2}, \tag{3.11}
$$

where v is the projection of p onto $\Gamma^{1,1}(\frac{1}{2}) \oplus E_8(-\frac{1}{2})$.

Let us now consider the fermionic sector in detail. The fermions in the $\Gamma^{1,1}_s$ lattice do not change under the $\mathbb{Z}_2$ symmetry, so together with the fermions in the uncompactified directions we have

$$
Z^f_{\Gamma^{1,1}_s}[{}^a_b] = \left(\frac{\vartheta[{}^a_b](\tau)}{\eta(\tau)}\right)^{3/2}. \tag{3.12}
$$

The orbifold blocks for two complex fermions with symmetry $\psi \to -\psi$ are given by (see for example [32], eq. (12.4.15)):

$$
\frac{\vartheta[{}^{a+h}_{b+g}](\tau)\vartheta[{}^{a-h}_{b-g}](\tau)}{\eta^2} \tag{3.13}
$$

Therefore, for the fermions in $\Gamma^{2,2} \oplus \Gamma^{1,1}_g$ one finds

$$
Z^f_{\Gamma^{2,2}\oplus\Gamma^{1,1}_g}[{}^h_g][{}^a_b] = \left(\frac{\vartheta[{}^{a+h}_{b+g}](\tau)\vartheta[{}^{a-h}_{b-g}](\tau)}{\eta^2}\right)^{3/4}. \tag{3.14}
$$

The treatment of the two fermions coming from $\Gamma^{1,9} \oplus \Gamma^{1,9}$ is slightly more delicate. The 00 block in the a,b sector is simply

$$
Z^f_{\Gamma^{1,9}\oplus\Gamma^{1,9}}[{}^0_0][{}^a_b] = \frac{\vartheta[{}^a_b](\tau)}{\eta(\tau)}. \tag{3.15}
$$

Let us now analyze the invariant states in the NS sector. A convenient basis for the Hilbert space $\mathcal{H}^{(1)}_{\rm NS} \otimes \mathcal{H}^{(2)}_{\rm NS}$ is given by

$$
\begin{aligned}
&\left(\psi^{(1)}_{-n_1}\cdots\psi^{(1)}_{-n_{2k}}\psi^{(2)}_{-m_1}\cdots\psi^{(2)}_{-m_l} \pm (1\leftrightarrow 2)\right)|0\rangle,\\
&\left(\psi^{(1)}_{-n_1}\cdots\psi^{(1)}_{-n_{2k+1}}\psi^{(2)}_{-m_1}\cdots\psi^{(2)}_{-m_{2l+1}} \mp (1\leftrightarrow 2)\right)|0\rangle,
\end{aligned}
\tag{3.16}
$$

where $n_i, m_i > 0$ are half-integers. The above states have the sign ±1, respectively, under the $\mathbb{Z}_2$ symmetry generator g which exchanges the two lattices. It is easy to see

that in computing the trace over the Hilbert space with an insertion of g, the above states cancel except when the (1) and the (2) content is the same. Therefore, only the states

$$
\begin{aligned}
&\psi^{(1)}_{-n_1}\cdots\psi^{(1)}_{-n_{2k+1}}\psi^{(2)}_{-n_1}\cdots\psi^{(2)}_{-n_{2k+1}}|0\rangle,\\
&\psi^{(1)}_{-n_1}\cdots\psi^{(1)}_{-n_{2k}}\psi^{(2)}_{-n_1}\cdots\psi^{(2)}_{-n_{2k}}|0\rangle
\end{aligned}
\tag{3.17}
$$

contribute to the trace, with signs -1 and $+1$ under g, respectively. An odd number of fermion oscillators leads to a -1 sign, but this is like having an insertion of $(-1)^F$. We then find

$$
\mathrm{Tr}_{\mathcal{H}^{(1)}_{\mathrm{NS}}\otimes\mathcal{H}^{(2)}_{\mathrm{NS}}} g\, q^{L_0-c/24} = \mathrm{Tr}_{\mathcal{H}_{NS}}(-1)^F\, q^{2L_0-c/12} = \left(\frac{\vartheta\left[{0 \atop 1}\right](2\tau)}{\eta(2\tau)}\right)^{\frac{1}{2}}, \tag{3.18}
$$

where the doubling in τ is due to the doubling in the oscillator content. Notice that the insertion of $(-1)^F$ in the above trace does not change anything, since $(-1)^{F_1}$ and $(-1)^{F_2}$ cancel each other, therefore

$$
Z^f_{\Gamma^{1,9}\oplus\Gamma^{1,9}}\left[{0 \atop 1}\right]\left[{a \atop b}\right] = \left(\frac{\vartheta\left[{a \atop 1}\right](2\tau)}{\eta(2\tau)}\right)^{\frac{1}{2}}, \tag{3.19}
$$

and the expressions for the other blocks can be obtained by modular transformations.

Putting all these results together, we can write up the one-loop partition functions for the different blocks. One finds, for example:

$$
Z\left[{0 \atop 0}\right] = \frac{1}{2\bar{\eta}^{24}(\tau)\eta^8(\tau)}\overline{\Theta}_{\Gamma_u}(\tau)\sum_{a,b}(-1)^{a+b+ab}\left(\frac{\vartheta\left[{a \atop b}\right](\tau)}{\eta(\tau)}\right)^4. \tag{3.20}
$$

for the 00 block. For the 01 block, one finds

$$
\begin{aligned}
Z\left[{0 \atop 1}\right] =& \frac{4}{\bar{\eta}^9(2\tau)\eta(2\tau)|\eta(\tau)|^3}\frac{1}{|\vartheta\left[{1 \atop 0}\right](\tau)|^3}\overline{\Theta}_{\Gamma_1}(\tau,\delta,0)\\
&\times\frac{\left(\vartheta\left[{0 \atop 0}\right](\tau)\right)^{3/2}\left(\vartheta\left[{0 \atop 1}\right](\tau)\right)^{3/2}}{\eta^3(\tau)}\frac{\left(\vartheta\left[{0 \atop 0}\right](2\tau)\right)^{1/2}-\left(\vartheta\left[{0 \atop 1}\right](2\tau)\right)^{1/2}}{\left(\eta(2\tau)\right)^{\frac{1}{2}}}.
\end{aligned}
\tag{3.21}
$$

3.2 The type II side of the FHSV model

The dual type II realization of the FSHV model is a compactification on the Enriques CY M with holonomy $SU(2)\times\mathbb{Z}_2$. The two covariant constant spinors of opposite chirality on M lead to $\mathcal{N}=2$ supersymmetry in four dimensions, but many features of the model are between the $SU(2)$ holonomy case with $\mathcal{N}=4$ supersymmetry and the generic situation with $SU(3)$ holonomy and $\mathcal{N}=2$ supersymmetry.

The compactification manifold M of the type II string is constructed as a free quotient of the manifold $Y=\mathrm{K3}\times\mathbb{T}^2$. The $\mathbb{Z}_2$ acts as the free Enriques involution [5] on the

K3 and as inversion $\mathbb{Z}_2 : z \mapsto -z$ on the coordinate z of the $\mathbb{T}^2$. If $\mathbb{T}^2 = \mathbb{C}/\mathbb{Z}^2$ is defined by $z \sim z + 1 \sim z + \tau$, we have four $\mathbb{Z}_2$ fixed points at $\{p_1, p_2, p_3, p_4\} = \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1}{2} + \frac{\tau}{2}\}$. The geometry of the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold is that of a $\mathbb{P}^1$ with area $\mathrm{Im}(\tau)/2$ and four conical curvature singularities at the p_i each of which has deficit angle π. The total space M is a K3 fibration over the $\mathbb{P}^1$, and by construction it has Enriques fibres E of multiplicity two, over the four p_i[3].

Every Enriques surface $E = \mathrm{K3}/\mathbb{Z}_2$ is a free quotient of a K3 by the Enriques involution $\rho : \mathrm{K3} \to \mathrm{K3}$. In order to construct a type II realization of the FHSV model, one first notices that the two-cohomology lattice $H^2(\mathrm{K3}, \mathbb{Z})$,

$$\Gamma_{\mathrm{K3}} = \Gamma^{3,19} = \Gamma_1^{1,9} \oplus \Gamma_2^{1,9} \oplus \Gamma_g^{1,1} \tag{3.22}$$

can be identified with the same blocks that appear in (3.1). The Enriques involution $\rho^*(p_1 \oplus p_2 \oplus p_5) = p_2 \oplus p_1 \oplus (-p_5)$ on Γ_{K3} acts as in (3.3). The $\Gamma_s^{1,1}$ lattice is spanned by $H^0(\mathrm{K3}, \mathbb{Z})$ and $H^4(\mathrm{K3}, \mathbb{Z})$ in the type II realization, and after quotienting by the involution ρ it can be identified as

$$\Gamma_s^{1,1} = H^0(E, \mathbb{Z}) \oplus H^4(E, \mathbb{Z}). \tag{3.23}$$

The shift on this lattice in the orbifold (3.3) corresponds to turning on a Wilson line expectation value for the RR $U(1)$ fields [4].

Some properties of the model are most clearly seen in an algebraic realization. We realize the two-torus as a hyperelliptic branched twofold covering of $\mathbb{P}^1$, with homogeneous coordinates denoted by $w : x$, and described by the equation

$$y^2 = f_4(w : x). \tag{3.24}$$

The $\mathbb{Z}_2$ acts as $\kappa : y \mapsto -y$, and the fixed points are the four branch points p_i of the degree four polynomial $f_4(w : x) = 0$. The holomorphic $(1, 0)$ form

$$\omega_{1,0} = \frac{\mathrm{d}x}{y} \tag{3.25}$$

is anti-invariant.

A similar realization of a K3 admitting the Enriques involution ρ is as a double covering of $\mathbb{P}^1 \times \mathbb{P}^1$ branched at the vanishing locus of a bidegree $(4, 4)$ hypersurface in $\mathbb{P}^1 \times \mathbb{P}^1$ [5, 28]. The total space is a eighteen-parameter family of K3 surfaces

$$\mathcal{Y}^2 = f_{4,4}(s : t, u : v)\ . \tag{3.26}$$

[3]It also exhibits an elliptic fibration over the Enriques E surface with four sections. An Enriques surface itself has two elliptic fibres with multiplicity two and can be obtained from dP_9 –a $\mathbb{P}^2$ blown up in nine points– by a logarithmic transform on two fibres.

The Enriques involution acts freely as

$$\rho : (\mathcal{Y}, s : t, u : v) \mapsto (-\mathcal{Y}, s : (-t), u : (-v)) \tag{3.27}$$

on a symmetric but otherwise generic slice of the family. The holomorphic $(2,0)$ form is given by

$$\omega_{2,0} = \frac{su\mathrm{d}t \wedge \mathrm{d}v}{\mathcal{Y}}. \tag{3.28}$$

Since ρ acts freely, the fundamental group of the Enriques surface E is $\mathbb{Z}_2$ and the Euler number is $\chi(E) = \chi(\mathrm{K3})/2 = 12$. As $\omega_{2,0}$ (and $\bar{\omega}_{2,0} = \omega_{0,2}$) is anti-invariant, the cohomology groups have dimensions $h^{00} = h^{22} = 1$, $h^{10} = h^{01} = h^{20} = h^{02} = 0$ and $h^{11} = \chi(E) - 2 = 10$. The canonical bundle is a two torsion class, i.e. $K_E^{\otimes 2} = \mathcal{O}_E$, hence non trivial: $K_E \neq \mathcal{O}_E$. On the blow up of the special configuration with

$$f_{4,4} = (u - v)(u + v)(as^4(u^2 - v^2) + bs^2t^2(u^2 - v^2) + t^4(cu^2 + dv^2)) \tag{3.29}$$

and with Picard number 18, one can explicitly check [5] that the invariant part Γ^+_{K3} and anti-invariant part Γ^-_{K3} of $\Gamma^{3,19}$ under ρ^* are

$$\Gamma^+_{\mathrm{K3}} = \Gamma^{1,1}(2) \oplus E_8(-2), \qquad \Gamma^-_{\mathrm{K3}} = [\Gamma^{1,1}(2) \oplus E_8(-2)] \oplus \Gamma^{1,1}_g. \tag{3.30}$$

The middle cohomology $H^2(E, \mathbb{Z})$ is isometric to the lattice

$$\Gamma_E = \frac{1}{2}\Gamma^+_{\mathrm{K3}} = \Gamma^{1,1}(1) \oplus E_8(-1). \tag{3.31}$$

The Calabi-Yau manifold M is constructed as $M = (\mathrm{K3} \times \mathbb{T}^2)/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ acts as

$$(\rho, \kappa) : (\mathcal{Y}, s : t, u : v, y, w : x) \mapsto (-\mathcal{Y}, s : -t, u : -v, -y : w : x). \tag{3.32}$$

On the generic K3 fiber the $\mathbb{Z}_2$ acts as the monodromy ρ, when the corresponding base point is transported in a loop around the special points p_i. The $\mathbb{Z}_2$ part of the holonomy is generated as follows. A tangent vector $v \in T_M$ transported over a nontrivial loop in the base (3.24) around two points of total deficit angle of π is inverted: $v \mapsto -v$. In the base direction the inversion occurs because of the deficit angles, and in the fiber direction due to the monodromy ρ.

The cohomology of M is easy to find. $\Omega = \omega_{2,0} \wedge \omega_{1,0}$ is invariant and becomes the unique, nowhere vanishing $(3,0)$-form on M. The 10 invariant $(1,1)$ forms $\omega^{(i)}_{1,1}$, $i = 1, \ldots, 10$ in Γ^+_{K3}, together with the volume form on $\mathbb{P}^1$, $\omega_{1,1}$, give 11 harmonic forms in $H^{1,1}(M, \mathbb{Z})$. We will adopt the type IIA interpretation in which the vector multiplets are mapped to the complexified Kähler moduli. Notice that the heterotic moduli of the Narain compactification are mapped to the Kähler moduli of the fiber (as we will make explicit in the next section), while the heterotic dilaton S is mapped

to the complexified Kähler modulus of the $\mathbb{P}^1$ base. Since $\chi(M) = 0$, one has $h^{2,1} = 11$. Ten of these forms can be explicitly constructed by taking the ten forms in Γ^-_{K3} of type $(1,1)$ and forming their wedge product with $\omega_{1,0}$. The remaining $(2,1)$ form is $\omega_{2,0} \wedge \omega_{0,1}$.

The moduli space of M has two different types of singular loci [18, 4] which lead to conformal field theories in four dimensions. The first degeneration comes from the shrinking of a smooth rational curve $e \in \Gamma_E$ with $e^2 = -2$. Since $\mathbb{P}^1$ has no unramified cover, the preimage of e in Γ_{K3} must be the sum $e_1 + e_2$ of two spheres e_1, e_2 in Γ_{K3} with $e_i \in \Gamma^{1,9}_i$ and $\rho^*(e_1) = e_2$. If e goes to zero size so do e_1 and e_2 in Γ_{K3}. The shrinking $\mathbb{P}^1$ leads to an $SU(2)$ gauge symmetry enhancement: in type IIA theory, a D2-brane wrapping the $\mathbb{P}^1$ with two possible orientations leads to massless $W^\pm$ bosons, which complete the corresponding $U(1)$ vector multiplet to a vector multiplet in the adjoint representation. This is plainly visible in the spectrum of the perturbative heterotic string, where the gauge group is realized by a level 2 WZW current. For each vanishing $e_1 + e_2$ in Γ^+_{K3} there is a vanishing $e_1 - e_2$ in the first summand of Γ^-_{K3} which leads to a hypermultiplet, also in the adjoint representation of the gauge group. We then obtain for this point the massless spectrum of $\mathcal{N} = 4$ supersymmetric gauge theory, which has a vanishing beta function and no Higgs branch.

The second degeneration is again plainly visible in the perturbative heterotic string and arises if one goes to the selfdual point in the lattice $\Gamma^{1,1}_s$ factor in (3.1). As usual one gets a $SU(2)$ gauge symmetry enhancement at level 1. In addition one gets four hypermultiplets in the fundamental representation of $SU(2)$, one from each fixed point of the $\mathbb{T}^2$. The resulting gauge theory is $\mathcal{N} = 2$, $SU(2)$ Yang-Mills theory with four massless hypermultiplets. This theory has a vanishing beta function and it is believed to be conformal [51]. It also has a Higgs branch which leads to a transition to a generic simply connected CY with $SU(3)$ holonomy and Hodge numbers $h_{21} = 10$ and $h_{11} = 16$ [18, 4].

An interesting difference between the two degenerations is that the first one occurs when a two-cycle of the covering K3 becomes small, while in the second one the full K3 surface has a volume of order the Planck scale [4].

The fact that these degenerations are associated to conformal theories indicates that there are no genus zero contributions to the Gromow-Witten invariants in type IIA theory on M [18]. For these degenerations, the Kähler class of the base is identified with the scale of the gauge coupling constant. An eventual scale dependence in $\mathcal{N} = 2$ supersymmetric theories comes from a one-loop correction to the beta function, which corresponds to worldsheet instantons with degree zero in the base, and from space time instantons, which are in turn related to the growth of worldsheet instantons with non-vanishing degree in the base. Both contributions are expected to vanish for the conformal theories. Later we will check with explicit computations that indeed there

are no worldsheet instanton corrections to the type IIA prepotential[4].

4 Heterotic computation of the F_g couplings

In this section we compute the couplings F_g in the heterotic side. It turns out that there are two natural lattice reductions to perform the computation: the geometric reduction, and the Borcherds-Harvey-Moore (BHM) reduction. We will present the results for the couplings in both reductions and we will also propose a type IIA interpretation of these results.

Before doing the lattice reduction, we have to evaluate the integrand (2.8) for the heterotic FHSV model. This is rather straightforward by using the results of the previous section. We have four orbifold blocks, but the first block (corresponding to $h = g = 0$) vanishes. The blocks $(h, g) = (0, 1), (1, 0), (1, 1)$ will be labelled by $J = 1, 2, 3$, and an easy computation shows that the modular forms $f_J(q)$ in (2.9) are given by

$$\begin{aligned} f_1(q) &= -\frac{128}{\eta^6(\tau)\vartheta_2^6(\tau)} = -\frac{2}{q}\prod_{n=1}^{\infty}(1-q^{2n})^{-12}, \\ f_2(q) &= \frac{4}{\eta^6(\tau)\vartheta_4^6(\tau)} = 4q^{-\frac{1}{4}}\prod_{n=1}^{\infty}(1-q^n)^{-12}(1-q^{n-1/2})^{-12}, \\ f_3(q) &= \frac{4}{\eta^6(\tau)\vartheta_3^6(\tau)} = 4q^{-\frac{1}{4}}\prod_{n=1}^{\infty}(1-q^n)^{-12}(1+q^{n-1/2})^{-12}. \end{aligned} \tag{4.1}$$

The Narain lattices for $J = 1, 2, 3$ are given in (3.8), and the corresponding theta functions in (2.9) are the same ones that appear in the computation of the one-loop partition function in the previous section. The modular forms in (4.1) have the right modular weight: the conjugate Narain-Siegel theta function for a lattice of signature $(2, 10)$ with $2g-2$ insertions has modular weight $(5, 2g-1)$, the modular form $\widehat{\mathcal{P}}_g$ has modular weight $(2g, 0)$, and the insertion τ_2^{2g-1} has modular weights $(-2g+1, -2g+1)$. Taking into account that the weight of the forms $f_J(q)$ is $(-6, 0)$, we see that the integrand in (2.8) has zero modular weight, as it should. Notice that the Narain-Siegel theta functions involve nonzero α, β and lattices which are not self-dual. This means that we will have to modify in an appropriate way the computation in [43]. We will now present the computation of the couplings in both reductions.

[4] A direct A-model argument can be given using relative Gromov-Witten invariants [45].

4.1 The geometric reduction

In order to apply the reduction technique we need explicit formulae for the projections of the lattice, that in turn depend on the moduli. Let us write a vector $p \in \Gamma_J$ as

$$p = (n, m, n_2, m_2, \vec{q}), \tag{4.2}$$

where (n, m), (n_2, m_2) are integer coordinates on $\Gamma_s^{1,1}$ and $\Gamma_d^{1,1}(\zeta_J)$, and $\vec{q}$ is a vector of integer coordinates in the E_8 lattice. The norm of p is given by

$$p^2 = p_R^2 - p_L^2 = 2nm + \zeta_J(2n_2m_2 - \vec{q}^{\,2}). \tag{4.3}$$

Since the lattices Γ_J have two $\Gamma^{1,1}$ factors, there are two natural reductions that one can use. The first one will be referred to as the geometric reduction. The reason for this name is that, as we will see, this reduction leads to an expression for F_g which is valid in the large volume limit of the Kähler moduli space and gives a generating functional of Gromov-Witten invariants, or equivalently, of BPS invariants that count D2-D0 bound states. In the geometric reduction, one chooses the reduction vector

$$z = (1, 0) \in \Gamma_s^{1,1}. \tag{4.4}$$

We then have $z' = (0, 1) \in \Gamma_s^{1,1}$. The reduced lattice is

$$K_J = E_8(-\zeta_J) \oplus \Gamma_d^{1,1}(\zeta_J), \qquad J = 1, 2, 3. \tag{4.5}$$

Different choices of reduction vectors correspond to different choices of cusps in the moduli space, and in particular lead to different parameterizations of this space. To make this explicit, we remind that the exact moduli space of vector multiplets for the Kähler parameters of the fiber is the coset $O(2, 10)/[O(2) \times O(10)]$. This coset is given by the following algebraic equations satisfied by the complex variables $(w_1, \cdots, w_{12})$ [19, 39]

$$\begin{aligned} \sum_{i=1}^{10} |w_i|^2 - |w_{11}|^2 - |w_{12}|^2 &= -\, 2Y, \\ \sum_{i=1}^{10} w_i^2 - w_{11}^2 - w_{12}^2 &= 0. \end{aligned} \tag{4.6}$$

The quantity Y above is the same one that appears in (2.9) and the mass formula gives

$$|p_R|^2 = \frac{|v \cdot w|^2}{Y}, \tag{4.7}$$

where the vector v is defined by

$$v = \left(\zeta_J^{\frac{1}{2}} \vec{q}, m - \frac{1}{2}n, \zeta_J^{\frac{1}{2}}(m_2 - \frac{1}{2}n_2), m + \frac{1}{2}n, \zeta_J^{\frac{1}{2}}(m_2 + \frac{1}{2}n_2)\right). \tag{4.8}$$

For the reduction vector (4.4) it is convenient to parameterize the coset by ten independent complex coordinates

$$y = (y^+, y^-, \vec{y}), \tag{4.9}$$

which are defined as follows

$$\begin{aligned} w_j =& y_j, \quad j = 1, \cdots, 8, \qquad w_9 = 1 + \frac{1}{4}y^2, \\ w_{10} =& \frac{1}{2}(y^+ - 2y^-), \qquad w_{11} = -1 + \frac{1}{4}y^2, \\ w_{12} =& \frac{1}{2}(y^+ + 2y^-), \end{aligned} \tag{4.10}$$

where the (complex) norm of the vector (4.9), y^2, is defined by

$$y^2 = 2y^+y^- - \vec{y}^{\,2}. \tag{4.11}$$

We will denote by $y_2^\pm$, $\vec{y}_2$ the imaginary parts of these moduli. From the above parameterization one finds,

$$Y = \big(\mathrm{Im}\, y\big)^2 = 2y_2^+ y_2^- - \vec{y}_2^{\,2}, \tag{4.12}$$

and $p_R = v \cdot w / Y^{\frac{1}{2}} \in \mathbb{C}$ is given by

$$p_R = \frac{1}{Y^{1/2}}\left(-n + \frac{1}{2}my^2 + \zeta_J^{\frac{1}{2}} m_2 y^+ + \zeta_J^{\frac{1}{2}} n_2 y^- + \zeta_J^{\frac{1}{2}} \vec{q} \cdot \vec{y}\right). \tag{4.13}$$

With this parameterization, the resulting topological couplings will have good convergence properties in the region $\mathrm{Im}\, y \to \infty$. Notice that

$$|z_+| = \frac{1}{Y^{\frac{1}{2}}}, \tag{4.14}$$

therefore $\nu = 1$. It is easy to evaluate the exponent of (2.20), which in this case it is equal to

$$2\pi \mathrm{i} r \cdot y = 2\pi \mathrm{i} \zeta_J^{\frac{1}{2}} \Big(m_2 y^+ + n_2 y^- + \vec{q} \cdot \vec{y}\Big), \tag{4.15}$$

We can now proceed to evaluate the integral (2.4). The first thing to observe is that, with the choice of reduction vector (4.4), only the untwisted sector $J = 1$ contributes. The reason for that is that in the twisted sectors $J = 2, 3$, the lattice $\Gamma_s^{1,1}$ where the reduction is performed has the shift $\beta_J = \delta = z - z'$. As shown in section 5 of [47], in those cases the integral over the fundamental domain is zero. This is easy to understand by looking at the expression (B.10) in Appendix B. The effect of this nonzero β is to shift $c \to c - 1/2$. As this integral is effectively evaluated when $|z_+| \to 0$, the integrand vanishes. On the other hand, the theta function associated to Γ_1 in the untwisted, projected sector $J = 1$ includes a phase $e^{\pi i \delta \cdot p}$. It was shown in [40] that the effect

of this phase is to shift the integer ℓ in (B.10) as $\ell \to \ell - \frac{1}{2}$. This means that the polylogarithm in (2.23) becomes

$$\mathrm{Li}_m(x) = \sum_{\ell=1}^{\infty} \frac{x^\ell}{\ell^m} \to \sum_{k=0}^{\infty} \frac{x^{k+\frac{1}{2}}}{(k+\frac{1}{2})^m} = 2^m \mathrm{Li}_m(x^{\frac{1}{2}}) - \mathrm{Li}_{\mathrm{m}}(x), \tag{4.16}$$

with $m = 3 - 2g$. In order to write the argument of the polylogarithm, we will relabel $m_2, n_2 \to m, n$, and introduce the moduli parameters $t = (t^+, t^-, \vec{t})$ as

$$-2\pi \mathrm{i} y = \sqrt{2} t. \tag{4.17}$$

The integer vector r in (4.15) reads

$$r = (n, m, \vec{q}) \in \Gamma_E = \Gamma^{1,1} \oplus E_8(-1), \tag{4.18}$$

and it follows from (3.31) that it labels two-cohomology classes in the Enriques fiber. It has norm $r^2 = 2mn - \vec{q}^{\,2}$. The argument of the polylogarithm is

$$x^{\frac{1}{2}} = \exp(-r \cdot t), \qquad r \cdot t = mt^+ + nt^- + \vec{q} \cdot \vec{t}.$$

The final formula for the F_g is then

$$F_g(t) = \sum_{r>0} c_g(r^2) \Big\{ 2^{3-2g} \mathrm{Li}_{3-2g}(\mathrm{e}^{-r\cdot t}) - \mathrm{Li}_{3-2g}(\mathrm{e}^{-2r\cdot t}) \Big\}, \tag{4.19}$$

where

$$\sum_n c_g(n) q^n = f_1(q) \mathcal{P}_g(q), \tag{4.20}$$

$f_1(q)$ is given in (4.1). The restriction $r > 0$ means that [24] $n > 0$, or $n = 0, m > 0$, or $n = m = 0$, $\vec{q} > 0$. The reason that the above formula involves $c_g(r^2)$ instead of $c_g(r^2/2)$ as in (2.23) is simply that the norm of the reduced lattice is twice the norm of the lattice in (4.18). Notice that due to the shift in ℓ there is no contribution from the "zero orbit" (B.12). The above expression is only valid in principle for $g > 0$, and the computation of the prepotential involves a somewhat different procedure explained in [24]. It is easy to check however that the worldsheet instanton corrections to the prepotential are given by (4.19) specialized to $g = 0$ (the same thing happens in the STU model analyzed in [43]). Since r^2 is always even and $f_1(q)$ has no even powers of q in its expansion, we conclude that the instanton corrections to F_0 vanish along the fiber directions. This is in agreement with the analysis of [18].

The genus one amplitude can be written as follows:

$$F_1(t) = -\frac{1}{2} \log \prod_{r>0} \left(\frac{1 - \mathrm{e}^{-r\cdot t}}{1 + \mathrm{e}^{-r\cdot t}} \right)^{2c_1(r^2)}. \tag{4.21}$$

The infinite product appearing in this equation was previously found by Borcherds in a related context [12]. In the Example 13.7 of that paper, Borcherds considers two different expressions for the same automorphic form, obtained by expanding it around different cusps. Both expressions are denominator formulae for two different superalgebras. The second denominator formula in this Example is precisely the infinite product of (4.21) (to see this one notices that the part of $2f_1(q)\mathcal{P}_1(q)$ involving even powers of q equals the modular form $f_{00}(2\tau)$ introduced by Borcherds).

We now propose the following type IIA interpretation of this computation. As shown in (3.23), the reduction lattice $\Gamma_s^{1,1}$ we are choosing here corresponds to the $H^0(E,\mathbb{Z})\oplus H^4(E,\mathbb{Z})$ cohomology of the Enriques surface, and z and z' are integer generators of $H^0(E,\mathbb{Z})$ and $H^4(E,\mathbb{Z})$, respectively. The remaining lattice can be identified to $H^2(E,\mathbb{Z})$, and indeed r in (4.18) is a set of integer coordinates for two-homology classes on the Enriques fiber. For this reduction, the region of moduli space where $\mathrm{Im}\, y \to \infty$ is the region where the D2s and the D0 are light (as shown in (4.13), their mass goes like $1/Y^{\frac{1}{2}}$ and $y/Y^{\frac{1}{2}}$, respectively) while the D4s are heavy (their masses go like $y^2/Y^{\frac{1}{2}}$). Therefore, the region $\mathrm{Im}\, y$ where this reduction is appropriate is the region of moduli space where the D2 and D0 are the lighter states, and the D4s wrapping the Enriques fiber are heavy. This is the large volume limit, and we expect the answer for F_g to encode information about Gromov-Witten invariants of the Enriques fiber. The sum over ℓ in (2.23) can be interpreted as the Poisson resummation of a sum over D0 brane charges (notice that ℓ appears after Poisson resummation of the integer n in (4.13)), and the shift in (4.16) corresponds to the RR Wilson line background along the $H^0(E,\mathbb{Z})$ direction [18, 4]. To substantiate this interpretation, we will see in the next section that (4.19) matches with the geometric computation of BPS invariants proposed in [31]. Moreover, in section 6 we will find perfect agreement of the heterotic predictions with a B-model computation of F_g for $g \leq 4$.

Although (4.19) is similar to other results obtained for heterotic models, it has some additional properties that make it particularly simple. For example, one can show that

$$C_{ij}\frac{\partial^2 F_1}{\partial t_i \partial t_j} = -16 F_2, \tag{4.22}$$

where C_{ij} is the intersection matrix of $\Gamma^{1,1}\oplus E_8(-1)$. This is a consequence of the following identity among the coefficients of the modular forms (4.20):

$$n\, c_1(n) = -4\, c_2(n), \quad n\,\mathrm{even}, \tag{4.23}$$

which can be proved by comparing the even part of the τ derivative of $f_1(q)\mathcal{P}_1(q)$ with the even part of $f_1(q)\mathcal{P}_2(q)$.

We now discuss the other possible lattice reduction available on the heterotic side.

4.2 The BHM reduction

Since the lattices (3.8) include the sublattice $\Gamma_d^{1,1}(\zeta_J)$, it is natural to compute the topological string amplitudes by choosing the reduction vector

$$z = (1,0) \in \Gamma_d^{1,1}(\zeta_J). \tag{4.24}$$

We call this the Borcherds-Harvey-Moore (BHM) reduction, since as we will see it is the choice of reduction made by Harvey and Moore in [26], and leads to the infinite product introduced by Borcherds in [11]. The reduced lattice is then

$$K_J = \Gamma_s^{1,1} \oplus E_8(-\zeta_J).$$

Since we have chosen a different reduction vector, the associated parameterization of the moduli space will be different from the one made in (4.10). We introduce, as before, ten independent complex coordinates $y = (y^+, y^-, \vec{y})$ defined through:

$$\begin{aligned} w_j =& y_j, \quad i = 1, \cdots, 8, \qquad w_9 = \frac{1}{2}(y^+ - 2y^-), \\ w_{10} =& 1 + \frac{1}{4}y^2, \qquad w_{11} = \frac{1}{2}(y^+ + 2y^-), \\ w_{12} =& -1 + \frac{1}{4}y^2. \end{aligned} \tag{4.25}$$

Although we have used the same notation for the y coordinates, they are related to the w coordinates in a different way than in the geometric reduction. With this parameterization, we find

$$p_R = \frac{1}{Y^{1/2}}\left(-\zeta_J^{\frac{1}{2}} n_2 + \frac{1}{2}\zeta_J^{\frac{1}{2}} m_2 y^2 + n y^- + m y^+ + \zeta_J^{\frac{1}{2}} \vec{q} \cdot \vec{y}\right). \tag{4.26}$$

Notice that the reduction vector has the norm

$$|z_+| = \left(\frac{\zeta_J}{Y}\right)^{\frac{1}{2}}, \tag{4.27}$$

hence the quantity ν introduced in (2.19) has the value $\nu = \zeta_J$ for each block. The exponent of (2.20) is now

$$\nu^{-\frac{1}{2}} 2\pi \mathrm{i} r \cdot y = 2\pi \mathrm{i} \zeta_J^{-\frac{1}{2}} \left(n y^- + m y^+ + \zeta_J^{\frac{1}{2}} \vec{q} \cdot \vec{y}\right). \tag{4.28}$$

The computation of the integral (2.8) is very similar to the one performed in [43], which we summarized in section 2 and Appendix B. The answer in this case is a sum over the three orbifold blocks $J = 1, 2, 3$, involving different lattices. One has now to be careful with the effects of the shifts α, β. Since with this choice of reduction vector

the shifts are orthogonal to z, z', they only lead to insertions of phases in the sum over the reduced lattice, as well as to shifts in their vectors, and their effect is easy to track. To write down the final answer, we first define the coefficients of modular forms:

$$\mathcal{P}_g(q) f_J(q) = \sum_n c_g^J(n) q^n. \tag{4.29}$$

Then, the couplings F_g are given by

$$\begin{aligned} F_g = {} & 2^{1-g} \sum_{r>0} (-1)^{m+n} c_g^1(mn - \vec{q}^{\,2}) \mathrm{Li}_{3-2g}(e^{-r\cdot t}) \\ & + 2^{g-1} \sum_{r>0} \left\{ c_g^2\left(\frac{1-\vec{q}^{\,2}}{4} + mn + \frac{m+n}{2} \right) - (-1)^{\vec{q}^{\,2}/2+m+n} c_g^3\left(\frac{1-\vec{q}^{\,2}}{4} + mn + \frac{m+n}{2} \right) \right\} \\ & \quad \cdot \mathrm{Li}_{3-2g}(e^{-r\star t}). \end{aligned} \tag{4.30}$$

In this equation, $r = (m, n, \vec{q})$, the coordinate $t = (t^{\pm}, \vec{t})$ is defined in terms of y by

$$-2\pi \mathrm{i} y = (\sqrt{2} t^{\pm}, \vec{t}), \tag{4.31}$$

and the inner products in (4.30) are given by

$$\begin{aligned} r \cdot t &= m t^+ + n t^- + \vec{q} \cdot \vec{t}, \\ r \star t &= (2m+1) t^+ + (2n+1) t^- + \vec{q} \cdot \vec{t}. \end{aligned} \tag{4.32}$$

The insertions $(-1)^{m+n}$ in the first and last block are due to the nonzero $\alpha = \delta$, while the shift in the second inner product in (4.32) is due to the shift by $\beta = \delta$, and we have relabelled $n \to n+1$. Finally, the insertion of $(-1)^{\vec{q}^{\,2}/2}$ in the $J = 3$ orbifold block comes from the insertion (3.11).

The expression (4.30) can be simplified as follows. First, one notices that $c_g^2(1/4 + p/2)$ equals $(-1)^{p+1} c_g^3(1/4 + p/2)$. This is easy to see by noting that they are the coefficients of modular forms related by $q^{\frac{1}{2}} \to -q^{\frac{1}{2}}$. Therefore, the $J = 2$ and $J = 3$ contributions are equal and add up. We will call their contribution the contribution of the twisted sector, while the contribution from $J = 1$ will be called the contribution of the untwisted sector. It is easy to see that the polylogarithms whose argument involves a Kähler class of the form $mt^+ + nt^- + \vec{q} \cdot \vec{t}$ with n and m *both odd* receive contributions from both the untwisted and twisted sector, while if m or n is even only the untwisted sector contributes. In the first case, the contributions come from the coefficients of *odd* powers in the modular form

$$2^{1-g} \mathcal{P}_g(q) f_1(q) + 2^g \mathcal{P}_g(q^4) f_2(q^4), \tag{4.33}$$

while in the second case they come from the contributions of *even* powers in the first term in (4.33). However, since the second term in (4.33) has only odd powers of q, we

can use the modular form (4.33) for both cases. As a last step, one notices by using doubling formulae (see Appendix A) that

$$\frac{64}{\eta^6(\tau)\vartheta_2^6(\tau)} = \frac{1}{\eta^6(4\tau)\vartheta_4^6(4\tau)}, \tag{4.34}$$

therefore $f_2(q^4) = -2f_1(q)$. We can finally write down a compact expression for F_g as follows:

$$F_g(t) = \sum_{r>0} c_g(r^2/2)(-1)^{n+m}\mathrm{Li}_{3-2g}(\mathrm{e}^{-r\cdot t}) \tag{4.35}$$

where the coefficients $c_g(n)$ are defined by

$$\sum_n c_g(n)q^n = f_1(q)\left\{2^{1-g}\mathcal{P}_g(q) - 2^{1+g}\mathcal{P}_g(q^4)\right\}, \tag{4.36}$$

and in (4.35) we regard r as a vector in $\Gamma^{1,1} \oplus E_8(-2)$, i.e. $r^2 = 2nm - 2\vec{q}^{\,2}$.

We now consider some particular cases of (4.35) in more detail. Although the above expression is in principle valid for $g \geq 1$, one can see again that the instanton corrections to the prepotential are given by its specialization to $g = 0$, and one finds $F_0 = 0$ due to a cancellation between the untwisted and twisted sectors. Let us now look at $g = 1$. This involves the modular form $\mathcal{P}_1(q)$ given in (2.15). The doubling formulae (A.12) gives

$$E_2(\tau) - 4E_2(4\tau) = -3\vartheta_3^4(2\tau), \tag{4.37}$$

and one finds

$$F_1 = -\frac{1}{2}\log\prod_{r>0}\left(1 - \mathrm{e}^{-r\cdot t}\right)^{(-1)^{n+m}c_{\mathrm{B}}(r^2/2)} \tag{4.38}$$

where

$$\sum_n c_{\mathrm{B}}(n)q^n = \frac{\eta(2\tau)^8}{\eta(\tau)^8\eta(4\tau)^8}. \tag{4.39}$$

This is the modular form introduced by Borcherds in [11], and the above expression for F_1 agrees with that found by Harvey and Moore in [26] (up to a factor of 1/2 due to different choice of normalizations). The infinite product appearing in (4.38) is the denominator formula of a superalgebra, and it was pointed out in [12] that it is actually identical to the infinite product in (4.21), but expanded around a different cusp. This is of course expected, since in both cases we are evaluating the same integral, but with different choices of reduction vector.

What is the interpretation of the F_g amplitudes in the BHM reduction? In the remaining of this subsection, we will make a proposal for what is the enumerative content of the topological string amplitudes in this reduction. The first thing to notice is that in the BHM reduction the reduced lattice is $H^0(E, \mathbb{Z}) \oplus H^4(E, \mathbb{Z})$ together

with the $E_8(-2)$ sublattice of $H^2(E,\mathbb{Z})$, and the integers n and m in (4.35) label zero and four-cohomology classes. The computation in this reduction is appropriate for the region $\mathrm{Im}\, y \to \infty$ in moduli space. However, one can see from (4.26) that this is the region where the light states are the D0, the D4 wrapping the Enriques surface, the D2s in $E_8(-2)$, and one of the D2s in $\Gamma^{1,1}_d$, while the other D2 in this sublattice (labelled by m_2 in (4.26)) is heavy. Therefore, we are *not* in the large radius regime, and the F_g amplitudes computed with this reduction do not have an enumerative interpretation in terms of Gromov-Witten theory. It is easy to see that they do not lead to integer GV invariants, and indeed we will see in the next sections that the usual Gromov-Witten/D2-D0 counting interpretation has to be reserved for the geometric reduction considered in the previous subsection. The F_g couplings in the BHM reduction must be counting bound states of the light states associated to this cusp. One hint about their BPS content comes from writing the generating functional as

$$\sum_{g=0}^{\infty}\sum_{r>0} c_g(r^2/2)q^{r^2/2}\lambda^{2g-2} = f_1(q)\left\{\frac{\xi^2(\lambda_+,q)}{4\sin^2(\lambda_+/2)} - 4\frac{\xi^2(\lambda_-,q^4)}{4\sin^2(\lambda_-/2)}\right\}, \tag{4.40}$$

where $\xi(\lambda,q)$ is given in (2.34), and

$$\lambda_\pm = \frac{\lambda}{2^{\pm\frac{1}{2}}}. \tag{4.41}$$

The derivation of (4.40) is very similar to the derivation of (2.33). This expression suggests that there should be a formula similar to (2.28), in which one does a Schwinger computation including the light states appropriate for this region of moduli space. The role of the D0s in the computation of [21] is now played by the light D2s in the reduction lattice. Since the BPS masses of these states have an extra factor $\nu^{\frac{1}{2}}$, the Schwinger integral in (2.28) becomes

$$\begin{aligned}&\int_0^\infty \frac{ds}{s}\left(2\sin\frac{s}{2}\right)^{2g-2}\sum_{n_2=-\infty}^{\infty}\exp\Big[-\frac{s}{\lambda}(r\cdot y+2\pi i\nu^{\frac{1}{2}}n_2)\Big] = \\ &\sum_{\ell=1}^{\infty}\frac{1}{\ell}\left(2\sin\frac{\ell\lambda}{2\nu^{\frac{1}{2}}}\right)^{2g-2}\exp\Big[-\frac{\ell}{\nu^{\frac{1}{2}}}r\cdot y\Big],\end{aligned} \tag{4.42}$$

where n_2 is the number of light D2 states. This fits with the heterotic expression given in (2.23), and has the effect of rescaling the string coupling constant by

$$\lambda \to \frac{\lambda}{\nu^{\frac{1}{2}}}, \tag{4.43}$$

which explains the overall factor ν^{1-g} in (2.23) as an effect of this rescaling, as well as the appearance of (4.41) in (4.40). In order for the logic of [21] to apply, however, it

g	$r_u^2/2=-1$	0	1	2	3	4	5	6	7	8
0	-2	0	-24	0	-180	0	-1040	0	-5070	0
1	0	4	12	64	172	576	1464	3840	9396	21056
2	0	0	-6	-32	-162	-576	-1980	-5760	-16470	-42112
3	0	0	0	8	60	336	1420	5280	17340	52640

Table 2: BPS invariants $(-1)^{n+m}n_g(r_{\rm u})$ for the untwisted sector, counting bound states of D0-D2-D4 branes on the Enriques fibre.

was important that the number of D2 branes bound to D0 branes is independent of the number of D0s. In our case, we would have a bound state problem involving D4, D2 and D0, and it is not clear that the number of bound states is independent of the number of D2s along the direction of the reduction vector z. On the other hand, we expect the number of BPS states to depend only on the norm of the vector of charges in the cohomology lattice of the fiber (4.3), as in the counting on K3 surfaces considered in [52, 25]. Since the D2 charge n_2 we are summing over in (4.42) is part of the $\Gamma_d^{1,1}$ sublattice, and we are setting $m_2=0$ in the complementary direction, the norm of the charge vector does not depend on n_2, therefore the degeneracies are independent of n_2 as in the situation considered in [21].

We then propose that (4.40) is a generating functional for an index that counts BPS particles keeping track of their helicities, as in [21]. These BPS particles are obtained from bound states of D4s wrapping the Enriques fiber, D2s wrapped around the curves in the E_8 sublattice of the Enriques cohomology, and D0s. As we argued above, a formula similar to (2.33) should hold after taking into account the rescaling of the string coupling constant (4.43) and the fact that we have two different sectors (untwisted and twisted). The natural labels for D-brane charges in the untwisted and twisted sector are, respectively,

$$r_{\rm u}=(n,m,\vec{q}), \qquad r_{\rm t}=(2m+1,2n+1,\vec{q}). \tag{4.44}$$

The difference between the D-brane charges in the two sectors is due to the fact that the D-branes in the untwisted sector are double covers of the D-branes in the twisted sector. For example [13], the twisted sector contains 4 D4 branes which have half the charge of a D4 brane in the untwisted sector. These "fractional" branes differ in their torsion charge. Bound states in the twisted sector are made out of one of these "fractional" D4 branes together with an arbitrary number of D4 branes with integer charge. This is why the D4-brane charge in the twisted sector is of the form $2m+1$. In order to count the bound states, we introduce two sets of BPS invariants for the twisted and untwisted sectors, $n_g(r_{\rm u})$ and $n_g(r_{\rm t})$. Our proposal for their generating

g	$r_t^2/2=-1$	1	3	5	7	9	11
0	8	96	720	4160	20280	87264	340912
1	0	0	-16	-192	-1488	-8896	-44944
2	0	0	0	0	24	288	2288
3	0	0	0	0	0	0	-32

Table 3: BPS invariants $(-1)^{n+m} n_g(r_{\rm t})$ for the twisted sector. Notice that $r_t^2/2$ is always odd, as follows from (4.44).

functionals is the natural one from the above results:

$$\begin{aligned}\sum_{r_{\rm u}} (-1)^{n+m} n_g(r_{\rm u}) z^g q^{r_{\rm u}^2/2} =& f_1(q)\xi^2(z,q),\\ \sum_{r_{\rm t}} (-1)^{n+m} n_g(r_{\rm t}) z^g q^{r_{\rm t}^2/2} =& -4f_1(q)\xi^2(z,q^4),\end{aligned} \tag{4.45}$$

where the norm of the vectors are computed as before in the lattice $\Gamma^{1,1}\oplus E_8(-2)$. Notice that the extra factor of 4 in the twisted sector corresponds to the D-brane states that differ in torsion classes. This factor is in turn related to the four hypermultiplets in the fundamental representation of $SU(2)$ that appear in the $N_f=4$ degeneration of the type IIA theory. We show some values of these BPS invariants in tables 2 and 3.

5 Geometric computation of the BPS invariants in the fibre

In this section we will analyze the heterotic predictions for the F_g amplitudes in the geometric reduction. We will extract the GV invariants and show that they fit with the geometrical approach developed in [31]. This will support our interpretation of the geometric reduction as the one corresponding to the counting of D2/D0 bound states.

As we already mentioned in section 2, the free energy of the perturbative topological string can be written in terms of BPS or GV invariants $n_g(r)$. The expansion (2.29) implies that the BPS invariants of non-constant maps $r\neq 0$ contribute to F_g as

$$F_g = \sum_r \mathrm{Li}_{3-2g}(\mathrm{e}^{-r\cdot t}) \sum_{h=0}^{g} a_{h,g} n_h(r)\ , \tag{5.1}$$

with $a_{g,g}=1, a_{g-1,g}=-(g-2)/12, \ldots, a_{2,g}=2(-1)^g/(2g-2)!, a_{1,g}=0, a_{0,g}=|B_{2g}|/(2g(2g-2)!)$. With (5.1) we rewrite now (4.19) in terms of the BPS invariants. In the type II picture they are expected to correspond to an integer index in the cohomology of the moduli space of D2 branes wrapping curves in the fiber direction of M. Let us now extract these invariants from (4.19).

g	$r^2=0$	2	4	6	8	10	12	14	16
0	0	0	0	0	0	0	0	0	0
1	8	128	1152	7680	42112	200448	855552	3345408	12166272
2	0	-16	-288	-2880	-21056	-125280	-641664	-2927232	-12166272
3	0	0	24	480	5264	41760	267360	1463616	7096992
4	0	0	0	-32	-704	-8400	-71872	-492800	-2872512
5	0	0	0	0	40	960	12384	113728	831960
6	0	0	0	0	0	-48	-1248	-17312	-169920
7	0	0	0	0	0	0	56	1568	23280
8	0	0	0	0	0	0	0	-64	-1920
9	0	0	0	0	0	0	0	0	72

Table 4: BPS invariants $n_g^{\rm odd}(r) \in \mathbb{Z}$ for the odd classes r in the fiber direction.

If at least one entry in $r \in \Gamma_E = \Gamma^{1,1} \oplus E_8(-1)$ is odd, the second term in (4.19) does not contribute and we get the invariants $n_g^{\rm odd}(r)$ listed in the table 4. Note that $r^2 \in 2\mathbb{Z}$ because Γ_E is even. In particular the prediction that $n_{g=0}^{\rm odd}(r) = 0$ follows from the fact that the modular form $f_1(q)$ in (4.1) has no even powers. The fact that all $n_g^{\rm odd}(r)$ are integers is an important check on the consistency of the calculation.

If all entries in r are even then $r^2 \in 8\mathbb{Z}$ and we call the class r even. In (4.19) the second term gives a subleading correction to $n_g^{\rm even}(r)$, and we again find that all of them are integer. The first few are listed in table 5.

In [21] the numbers $n_g(r)$ were given a geometrical interpretation. In simple cases they can be computed as an index on the cohomology $H^*(\mathcal{M})$ of the moduli space $\mathcal{M}$ of D2 branes [31]. By "simple" we mean that the D2 wraps an irreducible and possibly mildly nodal curve $C \in M$ in the class r. Infinitesimally, the moduli space is parameterized by the zero modes on the D2 brane. These form a supersymmetric spectrum with $2g$ zero modes of the flat $U(1)$ connection on C parameterizing the Jacobian $\mathrm{Jac}(C) \sim \mathbb{T}^{2g}$. Furthermore, there are $h^0(\mathcal{O}(C))$ zero modes corresponding to the deformations $\mathcal{M}_C$ of the curve C. For this reason the total moduli space $\mathcal{M}$ is expected to have a fibration structure

$$\begin{array}{ccc} \mathrm{Jac}(C) & \longrightarrow & \mathcal{M} \\ & & \downarrow \\ & & \mathcal{M}_C . \end{array} \tag{5.2}$$

The cohomology $H^*(\mathcal{M})$ has a natural $\mathrm{su}(2)_L \times \mathrm{su}(2)_R$ Lefshetz action which corresponds to the spacetime helicities of BPS bound states. The $\mathrm{su}(2)_R$ is essentially generated by the Kähler form of the base and the $\mathrm{su}(2)_L$ by the one of the fibre. The dimension of the cohomology group with eigenvalues j_L^3, j_R^3, $N_{j_L^3,j_R^3}(r)$, is not invariant under complex structure deformations. However, the index $n_g(r)$ defined in (2.26) is

g	$r^2=0$	8	16	24	32
0	0	0	0	0	0
1	4	42048	12165696	1242726144	69636018752
2	0	-21024	-12165696	-1864089216	-139272037504
3	0	5256	7096656	1708748448	174090046880
4	0	-704	-2872416	-1158884992	-165915421248
5	0	40	831948	611668944	127601309256
6	0	0	-169920	-254819136	-80867605120
7	0	0	23280	83673040	42545564896
8	0	0	-1920	-21406464	-18592299200
9	0	0	72	4174920	6721882484
10	0	0	0	-598848	-1994908928
11	0	0	0	59472	480175264
12	0	0	0	-3648	-92117568
13	0	0	0	104	13732280
14	0	0	0	0	-1531072

Table 5: BPS invariants $n_g^{\rm even}(r) \in \mathbb{Z}$ for the even classes r in the fiber direction.

an invariant. In general, it is not clear how to define the Lefshetz actions on $\mathcal{M}$. However, in [31] the problem was bypassed by using the Abel-Jacobi map, and the following formula for the $n_g(r)$ was derived

$$n_{g-\delta}(r) = (-1)^{(\dim(\mathcal{M}_C)+\delta)} \sum_{p=0}^{\delta} b_{g-p,\delta-p} \chi(\mathcal{C}^{(p)}), \quad b_{g,k} := \frac{2}{k!} \prod_{i=0}^{k-1} (2g-(k+1)+i), \quad b_{g,0} := 1 \,. \tag{5.3}$$

Here, $\mathcal{C}^{(p)}$ is the moduli space of the curve C in the class r together with a choice of p points, which correspond to nodes of C. In particular $\mathcal{C}^{(0)} = \mathcal{M}_C$. In (5.3) δ is the number of nodes and the formula is applied as follows. In the simplest situation C is a smooth curve of genus g in the class r, then $\delta = 0$ and

$$n_g(r) = (-1)^{\dim(\mathcal{M}_C)} \chi(\mathcal{M}_C). \tag{5.4}$$

This can be understood directly as follows. If C is smooth the $\mathrm{Jac}(C)$ is non-degenerate and carries I_g as $\mathrm{su}(2)_L$ Lefshetz representation of the fibre. The sum over j_R^3 in (2.26) gives –up to sign— the Euler number of the base. If the contribution to $n_{g-\delta}(r)$ comes only from an irreducible curve with δ nodes we can calculate in certain situations $\chi(\mathcal{C}^{(p)})$ to obtain the BPS number.

We now apply these ideas to D2 branes wrapping curves C in the fibre of the Calabi-Yau manifold M of the FHSV model. The moduli space $\mathcal{M}_C$ factorizes for these curves into $\mathcal{M}_C(F)$, parameterizing movements of C in the fibre, and $\mathbb{P}^1$, parameterizing

movements of C over the base of M. Along this $\mathbb{P}^1$ direction and outside the p_i, the Jac(C) is constant. The $\mathbb{P}^1$ is therefore a component, whose contribution factors in (2.26). Moreover on this component the su$(2)_R$ Lefshetz action in (2.26) reduces its contribution to an integral over the Euler class $\int_{\mathbb{P}^1} e$. This integral localizes to the p_i. The relevant part of the D2 brane moduli space to curves C in the fiber hence localizes to curves which sit in the Enriques fibre.

It is therefore sufficient to consider curves in the four special Enriques fibres to explain the BPS numbers in the tables in Sec. 5. Let us first recall an important fact about curves in an Enriques surfaces. According to proposition 16.1 in [5], for every such C in the class r in the Kähler cone there is a second curve $C + K_E$ in the class r up to torsion with $|C + K_E| \neq \emptyset$ and $r^2 = [C]^2 = [C + K_E]^2$. So each curve in the Enriques fibre is effectively doubled. Since we have four fibers we expect that the numbers in tables 4 and 5 are divisible by eight, which is indeed the case. Let us now compute the moduli space of deformations $\mathcal{M}_C$ for smooth curves of genus g. By the adjunction formula, for a curve C in the class $r = [C]$ we have

$$2g - 2 = [C]^2 + [C][K_E] \ , \tag{5.5}$$

where the second term $[C][K_E] = 0$ on an Enriques surface. The moduli space of the curve C in a surface S is given by the projectivization of $\mathcal{M}_C = \mathbb{P}H^0(\mathcal{O}(C))$, and the dimension $h^0(\mathcal{O}(C))$ can be calculated using the Riemann-Roch theorem

$$\chi(\mathcal{O}(C)) = \frac{[C]^2 + [C][K_S]}{2} + \chi(\mathcal{O}(S)), \tag{5.6}$$

where $\chi(\mathcal{O}(C)) = \sum_{i=1}^n (-1)^i h^i(\mathcal{O}(C))$. For smooth curves in an Enriques surface, $H^1(\mathcal{O}(C)) = H^2(\mathcal{O}(C)) = 0$ and $\chi(\mathcal{O}(C)) = h^0(\mathcal{O}(C))$. Moreover from section 3.2 we know that $\chi(\mathcal{O}(E)) = \sum_{i=1}^2 h^{i,0}(E) = 1$, and combining that with (5.5,5.6) yields

$$\mathcal{M}_C = \mathbb{P}^{g-1} \ . \tag{5.7}$$

We apply now (5.3) and get, for smooth curves in the class r of genus $g = \frac{r^2}{2} + 1$,

$$n_g(r) = 8 \cdot (-1)^{\frac{r^2}{2}} \chi(\mathbb{P}^{\frac{r^2}{2}}) = 8 \cdot (-1)^{\frac{r^2}{2}} \left(\frac{r^2}{2} + 1 \right) \tag{5.8}$$

in agreement with table 4.

Let us now give a more detailed calculation involving the nodal curves. The task is to calculate the Euler numbers $\chi(\mathcal{C}^{(\delta)})$. If we force the smooth curve C to pass through δ given points in E, corresponding to the locations of the nodes, we impose δ linear constraints on its moduli space $\mathcal{M} = \mathbb{P}^{g-1}$. The moduli space of deformations is therefore reduced to $\mathbb{P}^{g-\delta-1}$. On the other hand we are free to choose the position of the

g	$r^2=0$	2	4	6	8	10	12	14	16
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	24	288	2160	12544	61608
2	0	0	0	0	0	0	0	-32	-384
3	0	0	0	0	0	0	0	0	0

Table 6: Differences between the heterotic BPS prediction in table 2 and the geometric BPS calculation using (5.3,5.10).

points, which are therefore part of the moduli space of the nodal curves. The freedom of choosing n-points on E is naively E^n. Since the points are undistinguishable one considers the orbifold $\mathrm{Sym}^n(E) = E^n/S_n$ by the symmetric group S_n. The relevant model for the moduli space of n points is the "free field" resolution [55] $\mathcal{M}_\delta = \mathrm{Hilb}^\delta(E)$ of this orbifold. The name comes from the fact that the Euler numbers of the resolved spaces are generated by a free field representation

$$\sum_{n=0}^{\infty} \chi(\mathcal{M}_n) q^n = \prod_{n=1}^{\infty} \left(\frac{1}{1-q^n} \right)^{\chi(E)} = 1 + 12q + 90q^2 + 520q^3 + 2535q^4 + \ldots \qquad (5.9)$$

This is special bosonic case of a formula [22], which gives the Poincaré polynomial of $\mathcal{M}_n$ in terms of bosonic and fermionic free fields. The reason that one needs only 12 bosons here is that E has only even cohomology. Since $\mathrm{Hilb}^\delta(E)$ fibers trivially over $\mathbb{P}^{g-\delta-1}$ we obtain

$$\chi(\mathcal{C}^{(\delta)}) = (g-\delta)\chi(\mathcal{M}_\delta). \qquad (5.10)$$

If we insert this result into (5.3) we reproduce immediately, and to a large extent, the heterotic predictions in table 4. The deviations between the two calculations are given in table 6. As we will see later, the heterotic predictions are in full agreement with the computation of the topological string amplitudes by using the B model, and the deviations recorded in table 6 are due to the fact that for reducible curves with many nodes one has to refine the computation of BPS invariants in (5.3) as explained in [31].

It is instructive to compare this situation with the K3 curve counting of [55]. Because of $\chi(\mathcal{O}(K3)) = 2$ we get in that case $\mathcal{M}_C = \mathbb{P}^g$, instead of (5.7). On K3 we can force up to g nodes to get rational curves. For the Enriques surface, the smooth moduli of a genus g curve is too small to allow generically for g nodes. This also explains the absence of genus zero invariants.

We close this section by noticing that the genus one BPS invariants for odd classes listed in table 4 have an interesting algebraic interpretation. As we pointed out after (4.21), $F_1(t)$ can be interpreted as the logarithm of a denominator formula of a super-

algebra. The genus one BPS invariants $c_1(r^2)$ are then multiplicities of the (super)root spaces of this superalgebra.

6 The B-model for an algebraic realization of the FHSV model

In this section we find an algebraic realization for the double cover of the Enriques CY, and we study it and its $\mathbb{Z}_2$ quotient using mirror symmetry. To simplify the analysis we define a "reduced" FHSV model by blowing down an E_8 part of the Picard lattice. The reduced model has only three Kähler moduli, and its mirror can be analyzed in detail in the context of the B-model. We show that the topological string of the reduced model can be solved in terms of modular forms through the arithmetic properties of the mirror map. Using the holomorphic anomaly equations of [8] we find explicit, closed expressions for the topological string amplitudes up to genus 4. In this section we will denote genus g amplitudes by $F^{(g)}$.

6.1 The geometric description of the Enriques Calabi-Yau

Here we describe the periods of the Enriques Calabi-Yau and we find an algebraic description of the double cover and its mirror.

6.1.1 Periods and prepotential

If M is a $d = \mathrm{dim}_{\mathbb{C}}$ dimensional CY manifold and $\omega_{d,0}$ is its unique holomorphic $(d,0)$-form, we may consider the quantities

$$W^{(\mathbf{k})} = \int_M \omega_{d,0} \partial^{\mathbf{k}} \omega_{d,0}, \tag{6.1}$$

where $\partial^{\mathbf{k}} = \partial_{z_1}^{k_1} \ldots \partial_{z_{\rho-2}}^{k_{\rho-2}}$ denotes derivatives w.r.t. to the complex moduli. If the order of the derivative operator is $\sum k_i = |\mathbf{k}|$ then Griffiths transversality [14] implies

$$W^{(\mathbf{k})} = 0 \ , \quad \text{if } |\mathbf{k}| < d \ . \tag{6.2}$$

In particular, for $d \in 2\mathbb{Z}$ we get from $W^{(0)} = 0$ an algebraic relation between the periods, while for $d = 3$ eqs. (6.2) lead to $\mathcal{N} = 2$ special geometry.

Let us describe first properties of the periods of the manifold $M = (\mathrm{K3} \times \mathbb{T}^2)/\mathbb{Z}_2$, which follow from the double cover. On the K3 covering of the Enriques surface we can choose a twelve-dimensional basis of two-forms, α_i, $i = 0, \ldots, \rho-1$, in the anti-invariant lattice Γ^-_{K3} and satisfying

$$\int_{\mathrm{K3}} \alpha_i \wedge \alpha_j = \eta_{ij}. \tag{6.3}$$

Here η_{ij} is the symmetric, even intersection form on Γ^-_{K3}. We consider families of K3 surfaces covering the Enriques surface where $\rho = 12$ is the number of anti-invariant transcendental cycles, i.e. we choose a polarization so that the dual cycles, with basis Γ^j such that $\int_{\Gamma^j} \alpha_i = \delta_i^j$, are transcendental. We have $\Gamma^i \cap \Gamma^j = \eta^{ij}$ with $\eta_{ij}\eta^{jk} = \delta_i^k$. The discussion below holds for general algebraic K3 surfaces with ρ transcendental cycles, and in particular for the reduced model for which $\rho = 4$.

The holomorphic $(2,0)$ form can be expanded as

$$\omega_{2,0} = \sum_{i=0}^{\rho-1} \hat{X}^i \alpha_i \qquad \text{with} \quad \hat{X}^i = \int_{\Gamma^i} \omega_{2,0}. \tag{6.4}$$

The period integrals $\hat{X}^i(z)$ depend on the complex structure deformation parameters z_a, $a = 1, \ldots, \rho-2$, that appear in the algebraic definition of the model. Griffiths transversality (6.2) implies for $|\mathbf{k}| = 0$

$$\sum_{i,j=0}^{\rho-1} \hat{X}^i \hat{X}^j \eta_{ij} = 0. \tag{6.5}$$

and for $|\mathbf{k}| = 1$

$$\begin{aligned} \frac{3}{2}\partial_1 W^{(2,0,0)} = W^{(3,0,0)}, \quad \frac{1}{2}\partial_2 W^{(2,0,0)} + \partial_1 W^{(1,1,0)} = W^{(2,1,0)}, \\ \frac{1}{2}(\partial_1 W^{(0,1,1)} + \partial_2 W^{(1,0,1)} + \partial_3 W^{(1,1,0)}) = W^{(1,1,1)} \ . \end{aligned} \tag{6.6}$$

These can be integrated using the Picard-Fuchs (PF) equations and yield rational expressions for the B-model two-point function $W^{(\mathbf{k})}$, $|\mathbf{k}| = 2$ in terms of the complex structure deformation parameters z_k. We will denote them as $C_{z_i,z_j} := W^{(\mathbf{k})}$, $|\mathbf{k}| = 2$, where $\mathbf{k}$ has an entry 2 at the i'th position if $i = j$ and entries 1 at the i'th and j'th position otherwise.

There is a maximal unipotent point in the moduli space of complex structures of the K3 at which one has one unique holomorphic solution $\hat{X}^0$, $\rho - 2$ single logarithmic solutions $\hat{X}^a$, $a = 1, \ldots, \rho-2$ and one double logarithmic solution $\hat{F} := \hat{X}^{\rho-1}$. We define the mirror map for the K3 as

$$\hat{t}^a(z) = \frac{\hat{X}^a}{\hat{X}^0}(z) \qquad a = 1, \ldots, \rho-2. \tag{6.7}$$

The couplings C_{z_a,z_b} transform like sections of the bundle $\mathrm{Sym}^2 T^*\mathcal{M} \otimes \mathcal{L}^{-2}$, where $\mathcal{M}$ is the moduli space of complex structures on the K3 and $\mathcal{L}$ is the Kähler line bundle. The mirror map and the special gauge of the B-model w.r.t. to the Kähler line bundle relates then the B-model couplings to their A-model counterparts in the following way

$$C_{t^a t^b} = \frac{1}{(\hat{X}^0(z(t)))^2} C_{z_c z_d}(z(t)) \frac{\partial z_c}{\partial t^a}(z(t)) \frac{\partial z_d}{\partial t^b}(z(t)) = \hat{\eta}_{ab}, \tag{6.8}$$

where $\hat{\eta}_{ab}$ are the classical intersection numbers of the generators of the Picard lattice of the mirror K3, related to the η_{ij} above by

$$\eta_{ij} = h_{0,\rho-1} \oplus \hat{\eta}_{ab}, \qquad h_{0,\rho-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.9}$$

Eq. (6.8) is a consequence of (6.2) and reflects the absence of instanton corrections to the classical intersection ring of K3. It yields a simple relation between the quantity $\hat{X}^0(z(t))$, which corresponds to a gauge choice of $\omega_{2,0}$ in $\mathcal{L}^{-1}$, and the derivatives $\frac{\partial z_i}{\partial t_a}(z(t))$ of the mirror map. The latter is a total invariant under the subgroup Γ_X of the discrete automorphism group $\mathrm{Aut}(\Gamma^-_{K3})$, that is realized as monodromy on the periods $(\hat{X}^0, \ldots, \hat{X}^{\rho-1})$ on the algebraic family X double covering the Enriques surface E. The ability to express $\hat{X}^0(z(t))$ using (6.8) as a modular form of Γ_X will become important to solve for the $F^{(g)}$ in sec. 6.2. We can write the periods as a vector of inhomogeneous coordinates

$$\hat{\Pi} = \begin{pmatrix} \hat{X}^0 \\ \hat{X}^1 \\ \vdots \\ \hat{X}^{\rho-2} \\ \hat{X}^{\rho-1} \end{pmatrix} = \hat{X}^0 \begin{pmatrix} 1 \\ t^1 \\ \vdots \\ t^{\rho-2} \\ -\frac{1}{2}\hat{\eta}_{a,b} t^a t^b - 1 \end{pmatrix}, \tag{6.10}$$

where we used the mirror map (6.7) as well as (6.5) and the explicit form of η_{ij}. Note that the -1 in the last period is a specialization to $d = 2$ of the $\frac{X^0}{(2\pi i)^d}\zeta(d)\chi(M)$ term, which appears in $d = 3, 4$ CY manifolds.

Similarly we define for the $\mathbb{T}^2$ a basis of anti-invariant one forms α, β with $\int_{\mathbb{T}^2} \alpha \wedge \beta = 1$ and all other integrals are zero. We expand

$$\omega_{1,0} = x^0 \alpha - x^1 \beta \tag{6.11}$$

where the coefficients x^0, x^1 are the periods

$$x^0 = \int_a \omega_{1,0}, \qquad x^1 = \int_b \omega_{1,0} \tag{6.12}$$

and $\int_a \alpha = -\int_b \beta = 1$. At the point of maximal unipotent monodromy for the $\mathbb{T}^2$ we have a regular solution for x^0 and a logarithmic solution for x^1, and we define the mirror map as

$$\tau(z) = \frac{x^1}{x^0}(z). \tag{6.13}$$

With similar definitions as above we get a one-point function from integrating $\partial_1 W^{(1)} = W^{(2)}$. The analogue of (6.8) yields well known relations between the j-function and the Schwarz triangle functions for subgroups $\Gamma_{\mathbb{T}^2}$ of $SL(2,\mathbb{Z})$ (see, for example, [35]).

We combine information about $\mathbb{T}^2$ and K3 to write periods of M. The 3-cycles of M are

$$\begin{array}{llll} A_0 & = a \times \Gamma_0, & B_0 & = b \times \Gamma_{\rho-1}, \\ A_i & = a \times \Gamma_i, & B_i & = b \times \sum_{j=1}^{\rho-2} \eta_{ij} \Gamma_j, \qquad i = 1, \ldots \rho - 2, \\ A_\tau & = b \times \Gamma_0, & B_\tau & = a \times \Gamma_{\rho-1}. \end{array} \tag{6.14}$$

This basis is symplectic with $A_i \cap B_j = \delta_{ij}$. The invariant holomorphic $(3,0)$-form of M Ω is given by $\Omega = \omega_{2,0}\omega_{1,0}$ which in the algebraic model is realized as

$$\Omega = \frac{\mathrm{d}x}{y} \wedge su \frac{\mathrm{d}t \wedge \mathrm{d}v}{\mathcal{Y}}. \tag{6.15}$$

If we integrate this e.g. over A_0 we get

$$X^0 = \int_{A_0} \Omega = \int_a \left(\frac{\mathrm{d}x}{y} \right) \int_{\Gamma_0} \left(su \frac{\mathrm{d}t \wedge \mathrm{d}v}{\mathcal{Y}} \right) = x^0 \hat{X}^0 \ . \tag{6.16}$$

We can write the period vector of the threefold in a symplectic basis

$$\Pi = \begin{pmatrix} \int_{B_0} \Omega \\ \vdots \\ \int_{B_\tau} \Omega \\ \int_{A_0} \Omega \\ \vdots \\ \int_{A_\tau} \Omega \end{pmatrix} = \begin{pmatrix} -x^1 \hat{X}^{11} \\ -x^1 \eta_{1a} \hat{X}^a \\ \vdots \\ x^0 \hat{X}^{11} \\ x^0 \hat{X}^0 \\ x^0 \hat{X}^1 \\ \vdots \\ x^1 \hat{X}^0 \end{pmatrix} = X^0 \begin{pmatrix} 2\mathcal{F} - t^i \partial_i \mathcal{F} - \tau \partial_\tau \mathcal{F} \\ \partial_1 \mathcal{F} \\ \vdots \\ \partial_\tau \mathcal{F} \\ 1 \\ t^1 \\ \vdots \\ \tau \end{pmatrix} , \tag{6.17}$$

From the above we read off the prepotential

$$F_0 = -\frac{1}{2} \tau \hat{\eta}_{a,b} t^a t^b - \tau \tag{6.18}$$

and conclude that there are no instanton corrections at genus 0 in base, fibre and mixed directions.

$\Gamma_{\mathbb{T}^2}$ acts on the periods (x^0, x^1) a subgroup of $SL(2, \mathbb{Z})$. Similarly, Γ_X acts on $(\hat{X}^0, \ldots, \hat{X}^{\rho-1})$. The action of these two groups on Π does not commute and generates a bigger discrete group Γ_M. From the point of view of the heterotic string dual, Γ_X generates T-dualities, $\Gamma_{\mathbb{T}^2}$ generates S-dualities, and Γ_M is called the U-duality group.

6.1.2 Mirror symmetry on an algebraic realization

In order to calculate the periods discussed in the last section we need an algebraic realization and understand mirror symmetry on it. Let us first explain some features of

mirror symmetry for K3 surfaces. Mirror pairs of K3 can be given by three dimensional reflexive polyhedra following Batyrev's mirror symmetry construction [6]. Fortunately the double covering of Enriques that [28] uses is of this type. The small polyhedron Δ is given by the convex linear hull of the corners

$$\{\nu_1,\dots,\nu_5\} = \{[1,0,1],[0,1,1],[-1,0,1],[0,-1,1],[0,0,-1]\}\ . \tag{6.19}$$

Its points are on a lattice $\Lambda \sim \mathbb{Z}^3$, which makes it an integral polyhedron. Λ is obviously embedded like $\Lambda \in \Lambda_{\mathbb{R}} = N \sim \mathbb{R}^3$. The polyhedron is reflexive, which means that the dual $\Delta^* := \{x \in M^* | \langle x, y\rangle \geq -1, \forall\, y \in \Delta\}$ is an integral polyhedron in the dual lattice $\Lambda^* \in N^*$. The $\nu_1,\dots,\nu_5$ above are corners of the polyhedron, and the corners p_i^* of the large polyhedron Δ^* are found by solving the equations $\langle \nu_i^*, \nu_{j_k}\rangle = -1$, $k = 1,\dots,3$:

$$\{\nu_1^*,\dots,\nu_5^*\} = \{[2,2,1],[2,-2,1],[-2,-2,1],[-2,2,1],[0,0,-1]\}\ . \tag{6.20}$$

To the polyhedra Batyrev associate hypersurfaces

$$p = \sum_{i=1}^{\#\nu} a_i \prod_{j=1}^{\#\nu^*} x_j^{\langle \nu_i,\nu_j^*\rangle+1} = 0, \qquad p^* = \sum_{i=1}^{\#\nu^*} b_i \prod_{j=1}^{\#\nu} y_j^{\langle \nu_i,\nu_j^*\rangle+1} = 0, \tag{6.21}$$

which describe mirror manifolds M and W. For example we get a special double cover of $\mathbb{P}^1 \times \mathbb{P}^1$

$$p = a_1 s^4 u^4 + a_2 s^4 v^4 + a_3 t^4 v^4 + a_4 t^4 u^4 + a_5 y^2 + a_0 ystuv = 0\ . \tag{6.22}$$

Note that we take the product only over the corners of Δ^*, to which we associate the variables $\{x_j\} = \{s,u,t,v,y\}$, while we include all points that are not inside codimension one faces in Δ. Four of the a_i are redundant, i.e. they can be set to say one by the automorphism of the ambient space $\mathbb{P}_\Delta$.

Calabi-Yau threefolds correspond to 4d polyhedra and the cohomology groups of M and W are given by the formulas

$$\begin{aligned} h^{21}(M) &= l(\Delta) - 5 - \sum_{\operatorname{codim}(\theta)=1} l^*(\theta) + \sum_{\operatorname{codim}(\theta^*)=2} l^*(\theta^*) l^*(\theta), \\ h^{11}(M) &= l(\Delta^*) - 5 - \sum_{\operatorname{codim}(\theta^*)=1} l^*(\theta^*) + \sum_{\operatorname{codim}(\theta^*)=2} l^*(\theta^*) l^*(\theta)\ , \end{aligned} \tag{6.23}$$

where θ is a face (Δ is the top dim face) and $l(\theta)$ means all points in that face, while $l^*(\theta)$ means interior points in that face.

For K3 these formulas become

$$\begin{aligned} \rho(M) - 2 &= l(\Delta) - 4 - \sum_{\operatorname{codim}(\theta)=1} l^*(\theta) + \sum_{\operatorname{codim}(\theta)=2} l^*(\theta) l^*(\theta^*) \\ h(M) &= l(\Delta^*) - 4 - \sum_{\operatorname{codim}(\theta^*)=1} l(\theta^*) + \sum_{\operatorname{codim}(\theta)=2} l^*(\theta^*) l^*(\theta) \end{aligned} \tag{6.24}$$

and the interpretation is as follows. $h(M)$ is the rank of the Picard group of the K3 M and $\rho(M)$ is the number of transcendental cycles of M. For all 3d reflexive polyhedra one has $h(M) + \rho(M) = h^{11}(K3) + h^{20}(K_3) + h^{02} = 22$, and the different choices are merely different choices of the polarization. In our case we have $\rho(M) = 4$ and $h(M) = 18$. Here as above for the mirror W we just exchange Δ with Δ^*.

For the polyhedra Δ, i.e. the manifold M, we define the variables

$$z_1 = \frac{a_1 a_3 a_5^2}{a_0^4}, \qquad z_2 = \frac{a_2 a_4 a_5^2}{a_0^4}. \tag{6.25}$$

In terms of these variables we have the following Picard-Fuchs equations

$$\begin{aligned}\mathcal{L}_1 &= \theta_1^2 - 4(4\theta_1 + 4\theta_2 - 3)(4\theta_1 + 4\theta_2 - 1)z_1,\\ \mathcal{L}_2 &= \theta_2^2 - 4(4\theta_1 + 4\theta_2 - 3)(4\theta_1 + 4\theta_2 - 1)z_2,\end{aligned} \tag{6.26}$$

where we defined the logarithmic derivative $\theta_i = z_i \frac{\mathrm{d}}{\mathrm{d}z_i}$. The periods are linear solutions and in particular $z_i = 0$ is a point of maximal unipotency. Around this point we have a pure power series and two single logarithmic solutions, which correspond to geometric periods. In particular the mirror map is given by

$$\begin{aligned}z_1 &= q_1 - 40q_1^2 + 1324q_1^3 - 64q_1q_2 + 2560q_1^2q_2 + \mathcal{O}(q^4),\\ z_2 &= q_2 - 64q_1q_2 + 2560q_1^2q_2 - 40q_2^2 + 2560q_1q_2^2 + \mathcal{O}(q^4),\end{aligned} \tag{6.27}$$

with $q_i = \mathrm{e}^{-t_i}$, where t_i, $i = 1, 2$ are complexified Kähler parameters of the mirror. Indeed, calculation of the intersection numbers in the polyhedron Δ reveals that the corresponding Picard lattice is $\Gamma^{1,1}$. Using Griffiths transversality (6.2) and (6.26) we can evaluate the two-point functions

$$C_{x_1x_1} = \frac{2}{x_1\Delta}, \qquad C_{x_1x_2} = \frac{1 - x_1 - x_2}{x_1x_2\Delta}, \qquad C_{x_2x_2} = \frac{2}{x_2\Delta}\ . \tag{6.28}$$

Here

$$\Delta = 1 - 2(x_1 + x_2 + x_1x_2) + x_1^2 + x_2^2 \tag{6.29}$$

is the principal discriminant, and we rescaled the variables suitably $z_i = x_i/64$. Due to the above mentioned special properties of the mirror map of K3, we find that $C_{t_1t_1} = C_{t_2t_2} = 0$, $C_{t_1t_2} = 1$ in agreement with the identification of the Picard lattice $\Gamma^{1,1}$.

So far we have calculated on the mirror of the double covering of the Enriques surface and we need a justification that the two parameter family treated above does appear as a subfamily of the family of K3 admitting the Enriques involution. We consider now this symmetric splice of the mirror polynomial in (6.21). Let us label the coordinates $\{y_i\}$ again by $(s : t, u : v, y)$. The $\mathbb{Z}_2$ involution that we want to mod out $(s : t, u : v, y) \mapsto (-s : t, -u : v, -y)$ will break some of the automorphisms of the

ambient space. Therefore we can not follow Batyrev's methods that restricts to the true deformation parameters, which amounts to dropping the codimension one points, as used above.

The geometry of the mirror is again a double covering of $\mathbb{P}^1 \times \mathbb{P}^1$ branched at the generic degree $(4,4)$-hypersurface. The following expression keeps only monomials that are invariant under the $\mathbb{Z}_2$:

$$\begin{aligned} \mathcal{Y}^2 = \; & b_1 t^4 v^4 + b_2 t^4 u^2 v^2 + b_3 t^4 u^4 + b_4 u^4 t^2 s^2 + b_5 s^4 u^4 + b_6 s^4 u^2 v^2 \\ & + b_7 v^4 s^4 + b_8 s^4 t^2 s^2 + b_9 t^3 u^3 s v + b_{10} s^3 u^3 t v \\ & + b_{11} v^3 s^3 t u + b_{12} t^3 v^3 u s + b_{13} u^2 v^2 t^2 s^2 \end{aligned} \tag{6.30}$$

Normally the $SL(2,\mathbb{Z})$ transformations of the two $\mathbb{P}^1$ eliminate each three complex parameters. However to be compatible with the $\mathbb{Z}_2$ we can only make a rescaling $(s,t) \mapsto (\mu s, \mu^{-1} t)$ and similarly for (u,v). The weighted transformation of $y^2 \mapsto \mathcal{Y}^2 + \mathcal{Y} f_{2,2}(s,t,u,v)$ has been used to eliminate linear terms in y, and an overall rescaling will eliminate a third b_i. In total we have hence ten invariant b_i, which are precisely the deformation parameters of the Enriques surface.

It is still very tedious to derive a ten parameter PF system. Let us therefore look at a further symmetric restriction forced by the symmetry $u \mapsto iu$ and $s \mapsto is$ so that the $\mathbb{Z}_2 \times \mathbb{Z}_4$ invariant subslice has the form

$$b_0 \mathcal{Y}^2 = b_1 t^4 v^4 + b_3 t^4 u^4 + b_5 s^4 u^4 + b_7 v^4 s^4 + b_{13} u^2 v^2 t^2 s^2 \; . \tag{6.31}$$

We will argue now that this family is, up to a $\mathbb{Z}_2$ symmetry acting on their moduli space, identical to the family (6.22). At first glance this seems strange, because every monomial in the family (6.31) is invariant under the Enriques involution $(s:t, u:v, y) \mapsto (-s:t, -u:v, -y)$, while the monomial $ystuv$ is projected out from (6.22). We can keep this monomial by considering an induced action on the moduli space $a_0 \to -a_0$. Now recall, e.g. from the Landau-Ginzburg description of the CY manifold, that terms $\frac{\partial p}{\partial x_i}$ are trivial in the sense that they do not change the residua or periods in the compact case. We can hence use $\frac{\partial p}{\partial y} = 2a_5 y - a_0 stuv \sim 0$ and substitute $y = \frac{a_0}{2a_5} stuv$ is the last term of (6.22), which leads to the parameter identification $b_{13} = \left(\frac{a_0^2}{2a_5}\right)$, establishing the equivalence of the two families.

6.1.3 Arithmetic expressions for the B-model quantities

As discussed in sec. 6.1.1 we expect to be able to give arithmetic expressions for the mirror map and the fundamental period, similar to what is obtained in [34, 38]. If we restrict the PF system (6.26) to one variable by setting $z_2 = 0$ or $z_1 = 0$ we find the PF operator

$$\mathcal{L} = \theta^2 - 4(4\theta - 3)(4\theta - 1)z, \tag{6.32}$$

which corresponds to the $\Gamma(2)$ elliptic curve

$$x_1^2 = x_2^4 + x_3^4 + z^{-\frac{1}{4}} x_1 x_2 x_3. \tag{6.33}$$

After transforming this curve to the Weierstrass form we calculate its j-function as

$$j(q) = 1728 J(q) = \frac{(1+192z)^3}{z(1-64z)^2}. \tag{6.34}$$

Let us now define

$$\begin{aligned} K_2 &= \vartheta_3^4 + \vartheta_4^4, \\ K_4 &= \vartheta_2^8\ . \end{aligned} \tag{6.35}$$

In terms of these modular forms, the relation (6.34) can be inverted to obtain the Hauptmodul of $\Gamma(2)$ as

$$z(q) = \frac{K_4(q)}{64 K_2^2(q)}, \tag{6.36}$$

which is the arithmetic expression of the mirror map for the curve (6.33). The triviality of the one-point coupling for the elliptic curve

$$1 = \frac{1}{\omega_0^2} \frac{\mathrm{d}z}{\mathrm{d}t} \frac{1}{z(1-64z)}, \tag{6.37}$$

which is similar to the triviality of the two-couplings in the K3 case (6.8), leads to the following equation for the fundamental period ω_0 of the PF equation (6.32)

$$\omega_0^2(q) = \frac{1}{2} K_2(q)\ , \tag{6.38}$$

where we have used (6.36).

The full PF system (6.26) can also be solved arithmetically in terms of (6.36) by using the techniques of [38]. It can be easily shown that the simple ansatz

$$\begin{aligned} z_1(q_1,q_2) &= z(q_1)(1 - 64\, z(q_2)) = \frac{K_4}{64 K_2^2}\left(1 - \frac{\tilde{K}_4}{\tilde{K}_2^2}\right) \\ z_2(q_1,q_2) &= z(q_2)(1 - 64\, z(q_1)) = \frac{\tilde{K}_4}{64 \tilde{K}_2^2}\left(1 - \frac{K_4}{K_2^2}\right), \end{aligned} \tag{6.39}$$

where we defined $K_2 = K_2(q_1), K_4 = K_4(q_1)$ and $\tilde{K}_2 = K_2(q_2), \tilde{K}_4 = K_4(q_2)$, provides an analytic expression for the mirror map (6.27). One can also find an analytic expression for the fundamental period of the system (6.26) using (6.38)

$$(\hat{X}^0)^2(q_1,q_2) = \omega_0^2(q_1)\omega_0^2(q_2) = \frac{1}{4} K_2 \tilde{K}_2. \tag{6.40}$$

It is now easy to show that the discriminant (6.29) can be written as

$$\Delta = (1 - 64\, z(q_1) - 64\, z(q_2))^2, \tag{6.41}$$

where $z(q_i)$ is the mirror map (6.36).

In order to analyze the dependence of the model on the $\mathbb{T}^2$ in the base, it is convenient to realize it also algebraically, e.g. as degree 6 curve

$$x_1^6 + x_2^3 + x_3^2 + z^{-1/6} x_1 x_2 x_3 = 0 \tag{6.42}$$

in $\mathbb{P}(1,2,3)$, which also can be solved arithmetically. The mirror map is determined by the equation

$$J(q_3) = \frac{1}{z(q_3)(1 - 432 z(q_3))}, \tag{6.43}$$

which can be explicitly inverted to [35, 34]

$$z_3(q_3) = \frac{2}{J(q_3) + \sqrt{J(q_3)(J(q_3) - 1728)}}. \tag{6.44}$$

Finally, the fundamental period is

$$x^0(q_3) = E_4^{1/4}(q_3). \tag{6.45}$$

Therefore, the fundamental periods $x^0, \hat{X}^0$ as well as the mirror map for the reduced model can be expressed as modular forms in the parameters t_1, t_2, t_3 or functions of subgroups of $SL(2,\mathbb{Z})^3$, and this fact will become very useful in solving the B-model.

It is instructive to compare the reduced model with the Enriques Calabi-Yau constructed as an orbifold of $(\mathbb{T}_2)^3$ by $\mathbb{Z}_2^K \times \mathbb{Z}_2^E$, which acts of the coordinates of the three tori (z_1, z_2, z_3) as a Kummer involution and a free Enriques involution

$$\begin{array}{rllll} K: & - & , \quad - & , \quad + & \\ E: & +\left(\frac{1}{2}\right) & , \quad -\left(\frac{1}{2}\right) & , \quad - & \\ KE: & -\left(\frac{1}{2}\right) & , \quad +\left(\frac{1}{2}\right) & , \quad - & . \end{array} \tag{6.46}$$

Here $-$ indicates a sign change and $\left(\frac{1}{2}\right)$ a shift of the coordinate z_i. This model has three moduli from the invariant sector of the orbifold, which also have a natural $SL(2,\mathbb{Z})^3$ acting on them. However the geometry is very different. The rank 18 Picard lattice of the mirror of the $K3$ family X has intersection $E_8(-1) \oplus E_8(-1) \oplus H(1)$. As we will verify in detail in Sec. 6.2, the reduced model is obtained by contracting the curves in the $E_8(-1)$ part of the Picard lattice, after the Enriques identification. If we denote their complex volumes by t_i, $i = 1, \cdots, 8$, this is achieved by setting $t_i = 0$, i.e. $q_i = \mathrm{e}^{-t_i} = 1$. On the other hand the rank 18 Picard lattice of the Kummer K3 K that emerges after the $\mathbb{Z}_2^K$ orbifold in the first two coordinates is generated over

$\mathbb{Q}$ by sixteen $\mathbb{P}^1$'s that resolve the A_1 singularities at P_{ij}, $i = 1, \ldots, 4$, $j = 1, \ldots, 4$, and two invariant classes from the $\mathbb{T}_4$. If we call the resolution map $\sigma : K \to \mathbb{T}_4/\mathbb{Z}_2^K$ and the exceptional divisors $E_{ij} := \sigma^{-1}(P_{ij})$ we get $E_{ij} \cap E_{kl} = -2\delta_{ik}\delta_{jl}$. Obviously the lattices that get contracted to reach the orbifold point and the reduced model are quite different. Moreover due to the non-trivial B-field one expects that the complex volumes of the $\mathbb{P}^1$'s $\tilde{t}_i$ approach $|\tilde{t}_i| \to \pi \mathrm{i}$ at the orbifold limit so that $\tilde{q}_i = -1$. This might explain why all information about the full lattice disappears in a type II one-loop computation of F_1 in the invariant sector of the orbifold [7].

6.2 Topological string amplitudes from the reduced B-model

In this subsection we use the holomorphic anomaly equations of [8] to compute the topological string amplitudes for the reduced model in the fiber directions, up to (and including) $g = 4$.

6.2.1 Genus one amplitude

As explained in [8] the topological or holomorphic limit of the genus one free energy $F^{(1)}$ is related to the holomorphic Ray-Singer torsion [50]. The latter describes aspects of the spectrum of the Laplacians of $\Delta_{V,q} = \bar{\partial}_V \bar{\partial}_V^\dagger + \bar{\partial}_V^\dagger \bar{\partial}_V$ of a del-bar operator $\bar{\partial}_V$: $\wedge^q \bar{T}^* \otimes V \to \wedge^{q+1}\bar{T}^* \otimes V$ coupled to a holomorphic vector bundle V over M. More precisely with a regularized determinant over the non-zero mode spectrum of $\Delta_{V,q}$ one defines[5] [50]

$$I^{RS}(V) = \prod_{q=0}^{n} \left(\det{}' \Delta_{V,q}\right)^{\frac{q}{2}(-1)^{q+1}}. \tag{6.47}$$

One case of interest, $V = \wedge^p T^*$ with $\Delta_{p,q} := \Delta_{\wedge^p T^*,q}$, leads to the definition of a family index

$$F^{(1)} = \frac{1}{2} \log \prod_{p=0}^{n} \prod_{q=0}^{n} \left(\det{}' \Delta_{pq}\right)^{(-1)^{p+q} pq} \tag{6.48}$$

depending only on the complex structure of M. As was shown in [8] the holomorphic and antiholomorphic dependence of this object on the complex structure [9] can be integrated using special geometry to yield

$$F^{(1)} = \frac{1}{2} \log \left[\frac{f_1(z) \det\left(\frac{\partial z}{\partial t}\right)}{(X^0)^\kappa} \right]. \tag{6.49}$$

In this expression, X^0 is the fundamental period of the PF system, $\partial z / \partial t$ is the Jacobian of the mirror map, $\kappa = 3 + h_{11} - \frac{\chi}{12}$ depends on global topological data, and $f_1(z)$ is

[5] [49] reviews these facts and relates the Ray-Singer torsion to Hitchin's generalized 3-form action at one loop.

the holomorphic ambiguity at genus one. Up to the normalization factor $\frac{1}{2}$ this is the same expression that was derived in [7] using world-sheet arguments. The large volume behavior

$$F^{(1)} \to \sum_{i=1}^{h^{11}} t_i \int_M c_2(T) \wedge J_i, \qquad t \to \infty, \tag{6.50}$$

as well as local topological data of other singular limits in the complex structure moduli space, determine the leading behavior of $F^{(1)}$ and fix the holomorphic ambiguity $f(z)$.

We can now use our solution to the variations of Hodge structures for the family (6.22) to compute $F^{(1)}$ for this model. Indeed our variable choice at the point of maximal unipotent monodromy given in (6.25) is invariant under $a_0 \to -a_0$ and the result above can be used *verbatim* as describing a subfamily of the Enriques surface. Let us first calculate in the Enriques fibre direction from (6.49), which we parameterize as

$$F^{(1),E}(q_1, q_2) = r_1 \log \det \left(\frac{\partial z_i}{\partial t_j} \right) + r_2 \log(\hat{X}^0) + r_3 \log(z_1 z_2) + r_4 \log \Delta. \tag{6.51}$$

Notice that this can be also interpreted as a calculation for K3$\times\mathbb{T}^2$ in which one would expect that $F^{(1)}$ vanishes. This is the case if we set $r_1 = -r_2$, $r_2 = r_3 = r_4$. As we will explain in subsection 6.3.3, the heterotic prediction (4.19) is reproduced for the choice $r_1 = -\frac{1}{2}(2 + r_2)$, $r_3 = \frac{1}{2}(2 + r_2)$ and $r_4 = \frac{1}{4}(1 + r_2)$. Comparing with (6.49) we set $r_2 = -3$ to get $r_1 = \frac{1}{2}$. This yields also the expected result $\kappa = 3 + h_{11} - \frac{\chi}{12} = 6$ for the three parameter model.

We can now use the arithmetic expressions for the mirror map (6.39) and fundamental period (6.40) to derive an exact analytic B-model expression for the Ray-Singer torsion $F^{(1)E}(q_1, q_2) = F^{(1)}(q_1, q_2, q_3 = 0)$ in the Enriques direction (6.51)

$$F^{(1)\,E} = -\frac{1}{2} \log(\delta/16), \tag{6.52}$$

where

$$\delta = K_2^2 \tilde{K}_2^2 - K_4 \tilde{K}_2^2 - K_2^2 \tilde{K}_4 \tag{6.53}$$

and we used the fact that the discriminant (6.29) can be written as

$$\Delta = \left(\frac{\delta}{K_2^2 \tilde{K}_2^2} \right)^2. \tag{6.54}$$

Indeed, we can write as well

$$F^{(1)\,E} = -\frac{1}{4} \log(\Delta) - 2 \log(\hat{X}^0). \tag{6.55}$$

On the B-model side it is easy to argue that, in genus one, there are no contribution from curves in classes with mixed degree in base and fiber. According to (6.7,6.13)

the term $\det(\partial t/\partial z)$ will factorize. The same is true for the X^0 contribution due to (6.16). Finally the $f_1(z)$ is a product of discriminant factors. There will be two for the fibre and one for the base with no mixing between the base and fibre complex structure coordinates. Because of the logarithm we get a sum of two terms. The first one is the $F^{(1)E}$ in the fiber, and depends only on q_i, $i = 1, 2$ while the second one depends only of the τ parameter of the base, $q_3 = \exp(2\pi i\tau)$. We conclude that the total Ray-Singer torsion for the FHSV model is

$$F^{(1)}(q_1, q_2, q_3) = -\frac{1}{2}\log(\delta/16) - 12\log(\eta(q_3)), \tag{6.56}$$

where the contribution of the base is the same one as for $\mathrm{K3}\times\mathbb{T}^2$. This has also been argued in [26], however the expansion in [26] is not related to the instanton expansion of the type IIA string as explained in sections 4.1 and 4.2.

6.2.2 Propagators and higher genus amplitudes

In order to compute the $F^{(g)}$ amplitudes for $g > 1$ we use the holomorphic anomaly equations of [8]. These equations lead to an expression for $F^{(g)}$ of the form

$$F^{(g)} = F^{(g)}_{\rm rec} + (X^0)^{2g-2} f_g, \tag{6.57}$$

where $F^{(g)}_{\rm rec}$ is completely determined in a recursive way from the topological string amplitudes at lower genera $F^{(g')}$, $g' < g$, their derivatives w.r.t. the flat coordinates, and the propagators of the Kodaira-Spencer theory introduced in [8]. f_g is the holomorphic ambiguity at genus g. It is a rational function on the moduli space of complex structures and to determine it we need some extra data, like for example explicit values of Gromov-Witten invariants at low degree.

If we apply this procedure to the reduced, three–parameter model, we find important simplifications in the computation of $F^{(g)}_{\rm rec}$. This is due to the fact that, in flat coordinates, there is only one nonzero Yukawa coupling C_{123} which moreover does not receive any worldsheet instanton corrections and it is simply given by $C_{123} = 1$. Further derivatives of the Yukawa coupling vanish, and this sets to zero many contributions to $F^{(g)}_{\rm rec}$.

One of the fundamental ingredients of the holomorphic anomaly equations of [8] are the propagators S^{ij}, S^i and S of Kodaira-Spencer theory, where i, j are indices for the complex moduli. The procedure to find these propagators in the multiparameter case has been explained in [33]. It turns out that, in the case of the reduced model that we are studying, one can make a choice of gauge in which they have a particularly simple form. One first finds that it is possible to set $S^{ii} = 0$, $i = 1, 2, 3$. To solve for the remaining propagators with two indices S^{ij}, $i \neq j$, one can use the equation

$$\frac{1}{2}S^{ij}C_{ijk} = \partial_k F^{(1)} + \left(\frac{\chi}{24} - 1\right)\partial_k(\log X^0 + \log f), \tag{6.58}$$

where f is a holomorphic function of the complex moduli which arises as an integration constant of the holomorphic anomaly equations. The equation (6.58) determines uniquely the propagators S^{12}, S^{13}, S^{23} up to an integration constant. Using now (6.55), we see that the only piece of S^{13} that cannot be absorbed into f is $-\partial \log \hat{X}^0/\partial t_2$. The same thing happens to S^{23}, after exchanging $t_1 \leftrightarrow t_2$. We then make the following choice of propagators

$$\begin{aligned} S^{13} &= -\frac{1}{2}\frac{d}{dt_2}\log\, \tilde{K}_2 = \frac{1}{2}\partial_{t_2}F^{(1)} + A_2, \\ S^{23} &= -\frac{1}{2}\frac{d}{dt_1}\log\, K_2 = \frac{1}{2}\partial_{t_1}F^{(1)} + A_1, \end{aligned} \tag{6.59}$$

where

$$A_i = \frac{1}{8}\partial_{t_i}\log\,\Delta, \qquad i=1,2. \tag{6.60}$$

Finally, one can choose the remaining propagator S^{12} to be $-E_2(q_3)/12$. The final result for the propagators S^{ij} of the reduced model is then

$$\begin{aligned} S^{13} =& S^{31} = \frac{1}{12}E_2(q_2) + \frac{1}{8}\frac{\tilde{K}_4}{\tilde{K}_2} - \frac{1}{24}\tilde{K}_2, \\ S^{23} =& S^{32} = S^{13}(q_1 \leftrightarrow q_2), \\ S^{12} =& S^{21} = -\frac{1}{12}E_2(q_3), \\ S^{11} =& S^{22} = S^{33} = 0. \end{aligned} \tag{6.61}$$

Notice that, with this choice, S^{ij} only depends on q_k, where (ijk) is a permutation of 123. By using now the explicit expressions in [8], it is easy to see that the propagators S^i, S can be chosen to be

$$S^i = S^{ij}S^{ik}, \quad i=1,2,3, \qquad S = S^{12}S^{13}S^{23}, \tag{6.62}$$

where (ijk) is a permutation of 123. Notice that the structure of the propagators of the reduced model is very similar to the case of toroidal orbifolds studied in [8].

Using the one-loop result (6.56), the recursive formula of [8] for $F^{(2)}$ leads to the expression

$$\begin{aligned} F^{(2)} =& S^{12}(F^{(1)}_{12} + F^{(1)}_1 F^{(1)}_2) - 2S^{12}(S^{13}F^{(1)}_1 + S^{23}F^{(1)}_2) + 4S^{12}S^{13}S^{23} \\ &- \frac{1}{2}E_2(q_3)(S^{13}F^{(1)}_1 + S^{23}F^{(1)}_2 - 2S^{13}S^{23}) + (X^0)^2 f_2, \end{aligned} \tag{6.63}$$

where $F^{(1)}_i$ denote derivatives of $F^{(1)}$ w.r.t. t_i, $X^0 = \hat{X}^0(q_1,q_2)x^0(q_3)$, and f_2 is the holomorphic ambiguity. In the fibre limit

$$S^{12} \to -\frac{1}{12}, \quad X^0 \to \hat{X}^0, \quad f_2 \to f^E_2, \tag{6.64}$$

(6.63) should become $F^{(2)E}$, the genus two amplitude on the fiber. Here f_2^E is simply the fibre limit of the holomorphic ambiguity. The heterotic prediction can be recovered by simply setting

$$f_2^E = 0. \tag{6.65}$$

After using the explicit expressions for the propagators (6.61) and taking into account that

$$F_{12}^{(1)} = 8A_1A_2, \tag{6.66}$$

where the A_i are defined in (6.60), we find that the genus two free energy for the Enriques fibre is simply given by

$$F^{(2)\,E} = -\frac{1}{4}F_1^{(1)}F_2^{(1)}, \tag{6.67}$$

where $F_i^{(1)}$ can be written in terms of K_i, $\tilde{K}_i$, $E_2 = E_2(q_1)$ and $\tilde{E}_2 = E_2(q_2)$, as

$$F_i^{(1)} = \frac{1}{6}\Big(E_2(q_i) - \frac{\kappa K_2(q_i)}{2\delta}\Big), \quad i = 1, 2, \tag{6.68}$$

and $\kappa = \delta + 3K_4\tilde{K}_4$.

In the computation of $F^{(2)}$ we have used the formulae for $F_{\rm rec}^{(g)}$ obtained in [8], and we have applied them directly to the reduced model. However, for higher genus amplitudes this method is problematic. The reason for this is that, in order to obtain the correct result, we have to implement the reduction consistently, i.e. we have to first consider the formulae for $F_{\rm rec}^{(g)}$ in the original model with 11 Kähler parameters and then set the $E_8(-1)$ parameters to zero. In general, the result of this will be different from the result obtained by computing $F_{\rm rec}^{(g)}$ directly in the reduced, three–parameter model. The two procedures lead to the same tensorial structures, but with different numerical coefficients. It can be seen that at genus 2 this is not a problem, but for higher genus we cannot use the reduced model to obtain the answer for $F^{(g)}$.

One can still use the recursive formulae in the reduced model in order to obtain general properties of the amplitudes, as well as expressions for $F^{(g)E}$ in terms of modular forms. For $g = 3$ one finds, for example,

$$\begin{aligned} F^{(3)\,E} = &- \frac{1}{2^{10}3^4}\Bigg(\left(E_2^2 - \frac{\kappa E_2K_2}{\delta} + \frac{\rho_1}{(2\delta)^2}\right)\left(\tilde{E}_2^2 - \frac{\kappa\tilde{E}_2\tilde{K}_2}{\delta} + \frac{\rho_2}{(2\delta)^2}\right) \\ &+9\mu\left(\frac{1}{\delta^2}(E_2\tilde{K}_2 - \tilde{E}_2K_2)^2 - \frac{3}{4\delta^4}\rho_3\right)\Bigg), \end{aligned} \tag{6.69}$$

where $\mu = (K_2^2 - K_4)(\tilde{K}_2^2 - \tilde{K}_4)K_4\tilde{K}_4$ and

$$\begin{aligned} \rho_1 &= \kappa^2K_2^2 - 9(K_2^2 - K_4)K_4((\tilde{K}_2^2 + 7\tilde{K}_4)\delta + 9\tilde{K}_2^2K_4\tilde{K}_4), \\ \rho_2 &= \rho_1(q_1 \leftrightarrow q_2), \\ \rho_3 &= \delta^3 + (7\delta^2 + 72\delta K_4\tilde{K}_4)(K_2^2\tilde{K}_4 + K_4\tilde{K}_2^2) + 12K_4\tilde{K}_4(5\delta^2 + 6K_2^2\tilde{K}_2^2K_4\tilde{K}_4). \end{aligned} \tag{6.70}$$

m	$n=0$	1	2	3	4
0	-	8	4	8	4
1	8	2048	49152	614400	5373952
2	4	49152	5372928	216072192	5061451776
3	8	614400	216072192	21301241856	1063005978624
4	4	5373952	5061451776	1063005978624	100372720320512
5	8	37122048	83300614144	34065932304384	5641848336678912
6	4	216072192	1063005671424	794110053826560	218578429867425792

Table 7: Genus one BPS invariants $n_1(m,n)$ for the two parameter subfamily. By construction there is a symmetry under exchange of (m,n).

We have also found explicit results for the genus four topological string amplitude, which are too long to be reported here but are available on request. Note that $F^{(g)E}$ exhibits a leading order pole at the discriminant Δ of the form

$$F^{(g)E} \sim \frac{b_g}{\Delta^{g-1}} = \frac{\tilde{b}_g}{\delta^{2g-2}} \ , \tag{6.71}$$

which restricts the ansatz for the holomorphic ambiguity along the fiber f^E_g.

6.2.3 Comparison with the heterotic results

Let us now do a more detailed comparison of the B-model results with the heterotic predictions in the geometric reduction. Table 7 contains the B-model prediction for genus one BPS numbers in the classes (m,n) of $\Gamma^{1,1}$, the cohomology lattice of the reduced model. This is identified with the $\Gamma^{1,1}$ sublattice inside the Picard lattice of the Enriques surface $\Gamma_E = \Gamma^{1,1} \oplus E_8(-1)$. The total class in $\Gamma^{1,1} \oplus E_8(-1)$ is labelled by a vector $r = (n, m, \vec{v})$ with norm $r^2 = 2mn - \vec{v}^{\,2}$. The B-model for the subfamily calculates BPS invariants in which, for given (m,n), one sums over all possible vectors $\vec{v}$ in the $E_8(-1)$ lattice. Recall that the coefficients of q in the E_8-theta function

$$\Theta_{E_8}(q) = \sum_{\vec{v} \in E_8(1)} q^{\vec{v}^2/2} = \sum_{\vec{v}^2} m(\vec{v}^2/2) q^{\vec{v}^2/2} = E_4(q) = 1 + 240q + 2160q^2 + 6270q^3 + \ldots \tag{6.72}$$

yield the total number $m(\vec{v}^2/2)$ of vectors $\vec{v}$ with a given norm. This means that the results obtained from the reduced B model should correspond to a "reduced" heterotic theory in which we freeze the moduli of the $E_8(-1)$ lattice. Therefore, the topological string amplitudes of this "reduced" heterotic model will be given by

$$F^{\rm red}_g(t) = \sum_{n,m} c^{\rm red}_g(2nm) \Big\{ 2^{3-2g} {\rm Li}_{3-2g}(q_1^n q_2^m) - {\rm Li}_{3-2g}(q_1^{2n} q_2^{2m}) \Big\}, \tag{6.73}$$

where

$$\sum_n c_g^{\mathrm{red}}(n) q^n = f_1(q) E_4(2\tau) \mathcal{P}_g(q). \tag{6.74}$$

The first thing we notice when we compare the heterotic theory and the B-model is that, in the results from the B model, the BPS invariants depend only on the product nm but also on whether the class (n, m) is even or odd. For (m, n) in $\Gamma^{1,1}$ to be in a even class, m and n have to be even. In odd classes either m or n or both are odd. This is needed in order to match (6.73). One can easily see that indeed there is a precise agreement between (6.73) and the B-model results presented above. For example, the invariant 2048 at genus one in the B-model is given by

$$2048 = 128 + 240 \cdot 8, \tag{6.75}$$

where the number 240 counts the vectors of norm $\vec{v}^2 = 2$. Notice that, as a by-product of this comparison at $g = 1$, we obtain the following Borcherds-type identity

$$K_2^2(q_1)K_2^2(q_2) - K_4(q_1)K_2^2(q_2) - K_2^2(q_1)K_4(q_2) = 16 \prod_{n,m} \left(\frac{1 - q_1^n q_2^m}{1 + q_1^n q_2^m} \right)^{c(2nm)}, \tag{6.76}$$

where

$$\sum_n c(n) q^n = -\frac{64}{3\eta^6(\tau)\vartheta_2^6(\tau)} E_2(\tau) E_4(2\tau). \tag{6.77}$$

One can also check that the above B-model expressions on the fiber, for $2 \leq g \leq 4$, agree with the heterotic prediction for the reduced model (6.73).

6.2.4 Extending the results to the CY threefold

Let us now turn from the fibre limit to the full Enriques Calabi-Yau by including the base classes. We first notice one important general property of $F^{(g)}$: it will be a modular form with respect to a modular subgroup[6] in $\mathrm{SL}(2, \mathbb{Z}) \times \mathrm{SL}(2, \mathbb{Z}) \times \mathrm{SL}(2, \mathbb{Z})$, acting on the parameters (q_1, q_2, q_3). The modular weight is given by

$$(2g - 2, 2g - 2, 2g - 2). \tag{6.78}$$

This can be seen most easily by looking at the last term in (6.57). The holomorphic ambiguity has zero modular weight, since it is given by a rational function of the coordinates z_1, z_2, z, which are all modular forms of zero weight. The fundamental period X^0 has however modular weight $(1, 1, 1)$, which leads to (6.78).

[6] It was noticed in [54] that the $F^{(g)}$ calculated for the quintic in $\mathbb{P}^4$ in [31] can be written as polynomials of five generators. This is presumably a manifestation of modular properties of the correponding $F^{(g)}$ w.r.t. the modular group of the quintic in $SP(4, \mathbb{Z})$.

In order to study the full dependence on the $\mathbb{T}^2$ on the base, we first notice that the $F^{(g)}$ amplitudes restricted to the base vanish, since they should be equal to the topological string amplitudes of K3$\times\mathbb{T}^2$ on the base. If we denote $F^{(g)B}(q_3) = F^{(g)}(q_1 = 0, q_2 = 0, q_3)$, we have,

$$F^{(g)B} = 0, \qquad g > 1. \tag{6.79}$$

We now use this information to find expressions for $F^{(g)}(q_1, q_2, q_3)$. We first consider $g = 2$. Since $S^{13} = \mathcal{O}(q_2)$, $S^{23} = \mathcal{O}(q_1)$, the expression (6.63) reproduces the result (6.79) if we assume that the holomorphic ambiguity vanishes as well when restricted to the base. Given these facts, it is natural to assume that the total ambiguity f_2 vanishes for the choice of propagators (6.61). In this case, the only dependence of (6.63) on q_3 is an overall factor $E_2(q_3)$, and we finally obtain the remarkably simple expression for $g = 2$,

$$F^{(2)} = E_2(q_3)F^{(2)E}. \tag{6.80}$$

This simple form for the reduced model agrees with the results of Maulik and Pandharipande on the Gromov-Witten invariants at genus two for mixed classes of the full Enriques CY [45] (in fact, their results suggested to us the existence of a simple gauge for the propagators). Notice that (6.80) can be also written, after setting $q_3 = \mathrm{e}^{-t_3}$, as

$$F^{(2)} = -\frac{1}{2}F_1^{(1)}F_2^{(1)}F_3^{(1)}, \tag{6.81}$$

again with the same structure as the genus 2 amplitude for the toroidal orbifold considered in [8].

As is clear from (6.68), the expression (6.81) exhibits a pole at $\frac{1}{\Delta} \sim \frac{1}{\delta^2}$. A remarkable fact is that it does not have such a pole at the discriminant $\Delta_b = 1 - 432z_3$ of the base curve (6.42). It seems reasonable to assume that such poles do not occur at any genus and to refine (6.78) in that $F^{(g)}$ is a *holomorphic*, quasimodular function of q_3 of weight $2g-2$, i.e. in what concerns the dependence on q_3 it is generated by E_2, E_4, E_6. In this respect, the modular properties of $F^{(g)}$ with respect to the modular parameter of the torus would be similar to those found in the case of the elliptic curve [16]. On the other hand, $F^{(g)}$ is a *weakly holomorphic* function of q_1, q_2 with weights $(2g-2, 2g-2)$, which in particular contains δ^{-1}. This assumption restricts the ambiguity considerably and leads uniquely to the following expression for $F^{(3)}$:

$$F^{(3)} = E_2^2(q_3)F^{(3)E} + (E_2^2(q_3) - E_4(q_3))H^{(3)}(q_1, q_2), \tag{6.82}$$

where

$$H^{(3)}(q_1, q_2) = -\frac{1}{2}F^{(3)E} - \frac{1}{24}(F_1^{(1)E}F_2^{(2)E} + F_2^{(1)E}F_1^{(2)E}). \tag{6.83}$$

In the case $g = 3$ we don't have results on the mixed classes to compare with and check in detail the conjectures (6.82) and (6.83). However, we have verified that they lead

m	$n=0$	1	2	3	4
0	0	0	0	0	0
1	0	384	99072	2557440	34604544
2	0	99072	34604544	2425752576	82015423488
3	0	2557440	2425752576	399200753664	28156719273984
4	0	34604544	82015423488	28156719273984	3717898174470144

Table 8: Genus two BPS invariants $n_2(m,n,1)$ for branes wrapping the base torus once.

m	$n=0$	1	2	3	4
0	0	0	0	0	0
1	0	128	33792	10521600	17047552
2	0	33792	25704448	2596196352	113305067520
3	0	1052160	2596196352	635491780608	58963231506432
4	0	17047552	113305067520	58963231506432	10321183934611456

Table 9: Genus three BPS invariants $n_3(m,n,1)$ for branes wrapping the base torus once.

to results which are consistent with (6.79) and with integrality of the BPS numbers $n_g(r) \in \mathbb{Z}$ in the expansions (2.30). As expected from the discussion above (5.5), all of these numbers are divisible by eight. In the tables above we list some BPS invariants $n_g(r)$ for base degree d equal to one. In [23], we will extend these ideas and present an alternative and more powerful method to derive (6.82) and (6.83) which makes also possible to obtain results for $F^{(g)}$ to high genus.

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A Theta functions and modular forms

Our conventions for the Jacobi theta functions are:

$$\begin{aligned}
\vartheta_1(\nu|\tau) &= \vartheta[^1_1](\nu|\tau) = i\sum_{n\in\mathbf{Z}}(-1)^n q^{\frac{1}{2}(n+1/2)^2} e^{i\pi(2n+1)\nu},\\
\vartheta_2(\nu|\tau) &= \vartheta[^1_0](\nu|\tau) = \sum_{n\in\mathbf{Z}} q^{\frac{1}{2}(n+1/2)^2} e^{i\pi(2n+1)\nu},\\
\vartheta_3(\nu|\tau) &= \vartheta[^0_0](\nu|\tau) = \sum_{n\in\mathbf{Z}} q^{\frac{1}{2}n^2} e^{i\pi 2n\nu},\\
\vartheta_4(\nu|\tau) &= \vartheta[^0_1](\nu|\tau) = \sum_{n\in\mathbf{Z}}(-1)^n q^{\frac{1}{2}n^2} e^{i\pi 2n\nu},
\end{aligned} \tag{A.1}$$

where $q = e^{2\pi i\tau}$. When $\nu = 0$ we will simply denote $\vartheta_2(\tau) = \vartheta_2(0|\tau)$ (notice that $\vartheta_1(0|\tau) = 0$). The theta functions $\vartheta_2(\tau)$, $\vartheta_3(\tau)$ and $\vartheta_4(\tau)$ have the following product representation:

$$\begin{aligned}
\vartheta_2(\tau) &= 2q^{1/8}\prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2,\\
\vartheta_3(\tau) &= \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-\frac{1}{2}})^2,\\
\vartheta_4(\tau) &= \prod_{n=1}^{\infty}(1-q^n)(1-q^{n-\frac{1}{2}})^2
\end{aligned} \tag{A.2}$$

and under modular transformations they behave as:

$$\begin{aligned}
\vartheta_2(-1/\tau) &= \sqrt{\frac{\tau}{i}}\vartheta_4(\tau), & \vartheta_2(\tau+1) &= \mathrm{e}^{i\pi/4}\vartheta_2(\tau),\\
\vartheta_3(-1/\tau) &= \sqrt{\frac{\tau}{i}}\vartheta_3(\tau), & \vartheta_3(\tau+1) &= \vartheta_4(\tau),\\
\vartheta_4(-1/\tau) &= \sqrt{\frac{\tau}{i}}\vartheta_2(\tau), & \vartheta_4(\tau+1) &= \vartheta_3(\tau).
\end{aligned} \tag{A.3}$$

The theta function $\vartheta_1(\nu|\tau)$ has the product representation

$$\vartheta_1(\nu|\tau) = -2q^{\frac{1}{8}}\sin(\pi\nu)\prod_{n=1}^{\infty}(1-q^n)(1-2\cos(2\pi\nu)q^n+q^{2n}). \tag{A.4}$$

We also have the following useful identities:

$$\vartheta_3^4(\tau) = \vartheta_2^4(\tau) + \vartheta_4^4(\tau), \tag{A.5}$$

and

$$\vartheta_2(\tau)\vartheta_3(\tau)\vartheta_4(\tau) = 2\,\eta^3(\tau), \tag{A.6}$$

where

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \tag{A.7}$$

is the Dedekind eta function. One has the following doubling formulae,

$$\begin{aligned}
\eta(2\tau) =& \sqrt{\frac{\eta(\tau)\vartheta_2(\tau)}{2}}, \qquad \vartheta_2(2\tau) = \sqrt{\frac{\vartheta_3^2(\tau) - \vartheta_4^2(\tau)}{2}}, \\
\vartheta_3(2\tau) =& \sqrt{\frac{\vartheta_3^2(\tau) + \vartheta_4^2(\tau)}{2}}, \qquad \vartheta_4(2\tau) = \sqrt{\vartheta_3(\tau)\vartheta_4(\tau)}, \\
\eta(\tau/2) =& \sqrt{\eta(\tau)\vartheta_4(\tau)}.
\end{aligned} \tag{A.8}$$

The Eisenstein series are defined by

$$E_{2n}(q) = 1 - \frac{4n}{B_{2n}} \sum_{k=1}^{\infty} \frac{k^{2n-1} q^k}{1 - q^k}, \tag{A.9}$$

where B_m are the Bernoulli numbers. The formulae for the derivatives of the theta functions are also useful:

$$\begin{aligned}
q\frac{d}{dq} \log \vartheta_4 =& \frac{1}{24}\Big(E_2 - \vartheta_2^4 - \vartheta_3^4\Big), \\
q\frac{d}{dq} \log \vartheta_3 =& \frac{1}{24}\Big(E_2 + \vartheta_2^4 - \vartheta_3^4\Big), \\
q\frac{d}{dq} \log \vartheta_2 =& \frac{1}{24}\Big(E_2 + \vartheta_3^4 + \vartheta_4^4\Big),
\end{aligned} \tag{A.10}$$

and from these one finds,

$$q\frac{d}{dq} \log \eta = \frac{1}{24} E_2(\tau), \qquad q\frac{d}{dq} E_2 = \frac{1}{12}\Big(E_2^2 - E_4\Big). \tag{A.11}$$

The doubling formulae for $E_2(\tau), E_4(\tau)$ are

$$\begin{aligned}
E_2(2\tau) =& \frac{1}{2} E_2(\tau) + \frac{1}{4}(\vartheta_3^4(\tau) + \vartheta_4^4(\tau)), \\
E_4(2\tau) =& \frac{1}{16} E_4(\tau) + \frac{15}{16} \vartheta_3^4(\tau)\vartheta_4^4(\tau).
\end{aligned} \tag{A.12}$$

B Lattice reduction

In this Appendix we briefly review the computation of integrals of the form (2.8) with the technique of lattice reduction. This technique applies to integrals of the form

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_J f_J(\tau, \bar\tau) \overline{\Theta}_{\Gamma_J}(\tau, \alpha_J, \beta_J, P, \phi) \tag{B.1}$$

These integrals are sometimes called *theta transforms* of the (quasi)modular forms $f_J(\tau,\bar\tau)$. The generalized Siegel-Narain theta function which appears in this integral is defined as

$$\Theta_\Gamma(\tau,\alpha,\beta,P,\phi) =$$
$$\sum_{p\in\Gamma}\exp\left[-\frac{\Delta}{8\pi\tau_2}\right]\big(\phi(P(\lambda))\big)\exp\left[\pi i\tau(p+\beta/2)_+^2+\pi i\bar\tau(p+\beta/2)_-^2+\pi i(p+\beta/2,\alpha)\right], \tag{B.2}$$

where Γ is a lattice of signature (b^+,b^-), P is the projection, ϕ is a polynomial on $\mathbb{R}^{b^+,b^-}$ of degree m^+ in the first b^+ variables and of degree m^- in the second b^- variables, and Δ is the (Euclidean) Laplacian in $\mathbb{R}^{b^++b^-}$. The rest of the notations where introduced in section 2. The lattices involved in (B.1) have all the same signature, and they only differ in overall factors for their norms as well as in the shifts α_J,β_J. In the computation of these integrals by lattice reduction, one proceeds iteratively and in each step the rank of the lattice is reduced by two. Proceeding in this way, one can reduce the computation of (B.1) to evaluation of quantities associated to the reduced lattices.

Let us consider the simple case in which there is only one term in the sum (B.1), with $\alpha=\beta=0$, and the lattice Γ is even and self-dual. In this case we will denote (B.2) by $\Theta_\Gamma(\tau,P,\phi)$. The theta transform is then given by

$$\Phi_\Gamma(P,\phi,F^\Gamma) = \int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\overline{\Theta}_\Gamma(\tau,P,\phi)F^\Gamma(\tau,\bar\tau), \tag{B.3}$$

where

$$F^\Gamma(\tau,\bar\tau) = \tau_2^{b^+/2+m^+}F(\tau) \tag{B.4}$$

is a (quasi)modular form with weight $(-b^-/2-m^-,-b^+/2-m^+)$, constructed from the (quasi)modular form $F(\tau)$ with weights $(b^+/2+m^+-b^-/2-m^-,0)$. We will assume that $F(\tau)$ is an almost holomorphic form, *i.e.* it has the expansion

$$F(\tau) = \sum_{m\in\mathbf{Q}}\sum_{t\geq 0}c(m,t)q^m\tau_2^{-t} \tag{B.5}$$

where $c(m,t)$ are complex numbers which are zero for all but a finite number of values of t and for sufficiently small values of m. Lattice reduction is then implemented as follows. Let z be a primitive vector of Γ of zero norm, and let $K=(\Gamma\cap z^\perp)/\mathbb{Z}z$. This lattice, which has signature (b^+-1,b^--1), is called the reduced lattice. A typical situation when choosing a reduction vector occurs when the lattice Γ has $\Gamma^{1,1}$ as a sublattice, where $\Gamma^{1,1}$ is the lattice of rank two and intersection form

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}. \tag{B.6}$$

In this case, one can take z to be one of the vectors that generate $\Gamma^{1,1}$. In the reduced lattice one can construct "reduced" projections $\widetilde{P}$ as follows: consider $z_\pm \equiv P_\pm(z)$, and decompose $\mathbb{R}^{b^\pm} \simeq \langle z_\pm \rangle \oplus \langle z_\pm \rangle^\perp$. The projection on the orthogonal complement $\langle z_\pm \rangle^\perp$ is the reduced projection $\widetilde{P}_\pm$. It can be explicitly written in terms of $P_\pm$ as

$$\widetilde{P}_\pm(p) = P_\pm(p) - \frac{(P_\pm(p), z_\pm)}{z_\pm^2} z_\pm. \tag{B.7}$$

Once this reduced projection has been constructed, we have to decompose the polynomial involved in (B.2) with respect to this projection, according to the expansion

$$\phi(P(p)) = \sum_{h^+,h^-} (p, z_+)^{h^+} (p, z_-)^{h^-} \phi_{h^+,h^-}(\widetilde{P}(p)), \tag{B.8}$$

where p_{h^+,h^-} are homogeneous polynomials of degrees $(m^+ - h^+, m^- - h^-)$ on $\widetilde{P}(\Gamma \otimes \mathbb{R})$. We now write the vectors of the lattice Γ as

$$p = cz' + mz + p^K, \tag{B.9}$$

where p^K is a vector in the reduced lattice K and $(z', z) = N$. When the reduction vector belongs to a sublattice $\Gamma^{1,1}$, the vector z' is the other generator of the sublattice and $N = 1$. One can now rewrite the Siegel-Narain theta function in terms of the reduced lattice, after a Poisson resummation of the integer m, as [12]

$$\begin{aligned}\Theta_\Gamma(\tau, P, \phi) = {} & \frac{1}{\sqrt{2\tau_2 z_+^2}} \sum_{h \geq 0} \sum_{h^+,h^-} \frac{h!}{(-2i\tau_2)^{h^+ + h^-}} \left(-\frac{\tau_2 z_+^2}{\pi} \right)^h \binom{h^+}{h} \binom{h^-}{h} \\ & \times \sum_{c,\ell} (Nc\bar{\tau} + \ell)^{h^+ - h} (Nc\tau + \ell)^{h^- - h} \exp\left(-\frac{\pi |Nc\tau + \ell|^2}{2\tau_2 z_+^2} \right) \Theta_K(\tau, \ell\mu/N, -c\mu, \widetilde{P}, \phi_{h^+,h^-}).\end{aligned} \tag{B.10}$$

In writing this formula we have assumed that $(p_K, z') = (z', z') = 0$, and we have introduced the vector

$$\mu = -z' + N \left(\frac{z_+}{2z_+^2} + \frac{z_-}{2z_-^2} \right) \in K \otimes \mathbb{R}. \tag{B.11}$$

Once this expression is inserted into the integral (B.1), one can apply the "unfolding" procedure, in which the integral over the fundamental domain $\mathcal{F}$ of $\mathrm{SL}(2, \mathbb{Z})$ becomes an integral over the domain $[-1/2, 1/2] \times (0, \infty)$. At the same time, one can set $c = 0$ in (B.10) by modular invariance. There are two types of contributions in the end. The first one comes from $\ell = 0$ and it is sometimes called the contribution of the "zero orbit." It is given by

$$\frac{1}{\sqrt{2z_+^2}} \sum_{h \geq 0} \left(\frac{z_+^2}{4\pi} \right)^h \Phi_K(\widetilde{P}, \phi_{h,h}, F^K). \tag{B.12}$$

Notice that this is another theta transform, but for the reduced lattice, which is smaller. The contribution from the nonzero orbits comes from $\ell > 0$, and it involves a sum over the reduced lattice K. When $\widetilde{P}_+(\lambda^K) \neq 0$, it is given by

$$\begin{aligned}&\sqrt{\frac{2}{z_+^2}}\sum_{h\geq 0}\sum_{h^+,h^-}\frac{h!}{(2i)^{h^++h^-}}\left(-\frac{z_+^2}{\pi}\right)^h\binom{h^+}{h}\binom{h^-}{h}\sum_j\sum_{p^K\in K}\frac{1}{j!}\left(-\frac{\Delta}{8\pi}\right)^j\overline{p}_{h^+,h^-}(\widetilde{P}(p^K))\\&\cdot\sum_{\ell=1}^{\infty}\mathrm{e}^{2\pi i\ell(p^K,\mu)/N}\sum_t 2c(p^2/2,t)\left(\frac{\ell}{2|z_+||\widetilde{P}_+(p^K)|}\right)^{h-h^+-h^--j-t+b^+/2+m^+-3/2}\\&\cdot K_{h-h^+-h^--j-t+b^+/2+m^+-3/2}\left(\frac{2\pi\ell|\widetilde{P}_+(p^K)|}{|z_+|}\right).\end{aligned} \tag{B.13}$$

Here, $K_\nu(z)$ is the modified Bessel function, which comes from an integral over the strip $\tau_2 > 0$. When $\widetilde{P}_+(\lambda^K) = 0$, the integral has to be regularized and this leads to a slightly different expression which can be found in [12].

An important remark is that the above expressions are only valid if

$$|z_+|^2 \ll 1. \tag{B.14}$$

For a fixed primitive vector z, the value of z_+^2 depends on the projection we choose for the lattice, which is in turn determined by the Narain moduli of the string compactification. Different choices of regions in moduli space will require different choices of vectors for the lattice reduction, and therefore to different expressions for the resulting integral.

We have considered here the simplest case in which the theta transform (B.1) involves a single summand with $\alpha = \beta = 0$. More general cases can be also analyzed with the technique of lattice reduction, see [42, 40] for examples.

References

[1] M. Aganagic, A. Klemm, M. Mariño and C. Vafa, "The topological vertex," Commun. Math. Phys. **254**, 425 (2005) [arXiv:hep-th/0305132].

[2] I. Antoniadis, E. Gava, K. S. Narain and T. R. Taylor, "Topological amplitudes in string theory," Nucl. Phys. B **413**, 162 (1994) [arXiv:hep-th/9307158].

[3] I. Antoniadis, E. Gava, K. S. Narain and T. R. Taylor, "N=2 type II heterotic duality and higher derivative F terms," Nucl. Phys. B **455**, 109 (1995) [arXiv:hep-th/9507115].

[4] P. S. Aspinwall, "An N=2 Dual Pair and a Phase Transition," Nucl. Phys. B **460**, 57 (1996) [arXiv:hep-th/9510142].

[5] W. Barth, C. Peters and A. Van den Ven, *Compact Complex Surfaces*, Springer-Verlag, Berlin (1984).

[6] V. Batyrev, "Dual Polyhedra and Mirror Symmetry for Calabi-Yau Hypersurfaces in Toric Varieties," J. Algebraic Geom. **3** (1994) 493.

[7] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, "Holomorphic anomalies in topological field theories," Nucl. Phys. B **405**, 279 (1993) [arXiv:hep-th/9302103].

[8] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, "Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes," Commun. Math. Phys. **165**, 311 (1994) [arXiv:hep-th/9309140].

[9] J.-M. Bismut, H. Gillet and C. Soulé, "Analytic Torsion and Holomorphic Determinant Bundles I,II and III," Commun. Math. Phys. **115**, 49 (1988), Commun. Math. Phys. **115**, 79 (1988) and Commun. Math. Phys. **165**, 301 (1988).

[10] C. Borcea, "K3 surfaces with involutions and Mirror Pairs of Calabi-Yau manifolds," in Mirror Symmetry II, Ed. B. Greene and S.T. Yau, (1997) AMS/International Press.

[11] R. E. Borcherds, "The moduli space of Enriques surfaces and the fake monster Lie superalgebra," Topology **35**, 699 (1996).

[12] R. E. Borcherds, "Automorphic forms with singularities on Grassmannians," Invent. Math. **132**, 491 (1998) [arXiv:alg-geom/9609022].

[13] I. Brunner, J. Distler and R. Mahajan, "Return of the torsion D-branes," Adv. Theor. Math. Phys. **5**, 311 (2002) [arXiv:hep-th/0106262].

[14] R. Bryant and P. Griffith, "Some observations on the Infinitesimal Period Relations for Regular Threefolds with Trivial Canonical Bundle," Progr. Math. **36**, M. Artin and J. Tate, Eds. Birkhäuser, 1983.

[15] A. Dabholkar, F. Denef, G. W. Moore and B. Pioline, "Precision counting of small black holes," arXiv:hep-th/0507014.

[16] R. Dijkgraaf, "Mirror Symmetry and Elliptic Curves," and M. Kaneko and D. Zagier, "A generalized Jacobi Theta function and Quasimodular Forms," in *The Moduli Space of Curves*, Prog. Math. **129** (Birkhäuser, 1995), 149 and 165.

[17] L. J. Dixon, V. Kaplunovsky and J. Louis, "Moduli Dependence Of String Loop Corrections To Gauge Coupling Constants," Nucl. Phys. B **355**, 649 (1991).

[18] S. Ferrara, J. A. Harvey, A. Strominger and C. Vafa, "Second quantized mirror symmetry," Phys. Lett. B **361**, 59 (1995) [arXiv:hep-th/9505162].

[19] S. Ferrara, C. Kounnas, D. Lust and F. Zwirner, "Duality invariant partition functions and automorphic superpotentials for (2,2) string compactifications," Nucl. Phys. B **365**, 431 (1991).

[20] A. Gathmann, "Topological recursion relations on Gromov-Witten invariants of higher genus," math.AG/0305361.

[21] R. Gopakumar and C. Vafa, "M-theory and topological strings. I& II," [arXiv:hep-th/9809187] and [arXiv:hep-th/9812127].

[22] L. Göttsche and W. Soergel, "Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces," Math. Ann. **296** (1993) 235.

[23] T. Grimm, A. Klemm, M. Mariño and M. Weiss, to appear.

[24] J. A. Harvey and G. W. Moore, "Algebras, BPS States, and Strings," Nucl. Phys. B **463**, 315 (1996) [arXiv:hep-th/9510182].

[25] J. A. Harvey and G. W. Moore, "On the algebras of BPS states," Commun. Math. Phys. **197**, 489 (1998) [arXiv:hep-th/9609017].

[26] J. A. Harvey and G. W. Moore, "Exact gravitational threshold correction in the FHSV model," Phys. Rev. D **57**, 2329 (1998) [arXiv:hep-th/9611176].

[27] M. Henningson and G. W. Moore, "Threshold corrections in K(3) x T(2) heterotic string compactifications," Nucl. Phys. B **482**, 187 (1996) [arXiv:hep-th/9608145].

[28] E. Horikawa, "On the periods of Enriques Surfaces I& II," Math. Ann. **234** (1978) 73 and Math. Ann. **235** (1978) 217.

[29] S. Hosono, M. H. Saito and A. Takahashi, "Holomorphic anomaly equation and BPS state counting of rational elliptic surface," Adv. Theor. Math. Phys. **3**, 177 (1999) [arXiv:hep-th/9901151].

[30] S. Kachru and C. Vafa, "Exact results for N=2 compactifications of heterotic strings," Nucl. Phys. B **450**, 69 (1995) [arXiv:hep-th/9505105].

[31] S. Katz, A. Klemm and C. Vafa, "M-theory, topological strings and spinning black holes," Adv. Theor. Math. Phys. **3**, 1445 (1999) [arXiv:hep-th/9910181].

[32] E. Kiritsis, "Introduction to superstring theory," hep-th/9709062.

[33] A. Klemm, M. Kreuzer, E. Riegler and E. Scheidegger, "Topological string amplitudes, complete intersection Calabi-Yau spaces and threshold corrections," JHEP **0505**, 023 (2005) [arXiv:hep-th/0410018].

[34] A. Klemm, W. Lerche and P. Mayr, "K3 Fibrations and heterotic type II string duality," Phys. Lett. B **357**, 313 (1995) [arXiv:hep-th/9506112].

[35] A. Klemm, B. H. Lian, S. S. Roan and S. T. Yau, "A Note on ODEs from mirror symmetry," arXiv:hep-th/9407192.

[36] A. Klemm and M. G. Schmidt, "Orbifolds By Cyclic Permutations Of Tensor Product Conformal Field Theories," Phys. Lett. B **245**, 53 (1990).

[37] W. Lerche, B. E. W. Nilsson, A. N. Schellekens and N. P. Warner, "Anomaly Cancelling Terms From The Elliptic Genus," Nucl. Phys. B **299**, 91 (1988).

[38] B. H. Lian and S. T. Yau, "Arithmetic properties of mirror map and quantum coupling," Commun. Math. Phys. **176**, 163 (1996) [arXiv:hep-th/9411234]. "Mirror maps, modular relations and hypergeometric series 1, & 2" arXiv:hep-th/9507151 & Nucl. Phys. Proc. Suppl. **46**, 248 (1996) [arXiv:hep-th/9507153].

[39] G. Lopes Cardoso, D. Lust and T. Mohaupt, "Threshold corrections and symmetry enhancement in string compactifications," Nucl. Phys. B **450**, 115 (1995) [arXiv:hep-th/9412209].

[40] C. Lozano and M. Mariño, "Donaldson invariants of product ruled surfaces and two-dimensional gauge theories," Commun. Math. Phys. **220**, 231 (2001) [arXiv:hep-th/9907165].

[41] D. Lüst, "String vacua with N = 2 supersymmetry in four dimensions," arXiv:hep-th/9803072.

[42] M. Mariño and G. W. Moore, "Donaldson invariants for non-simply connected manifolds," Commun. Math. Phys. **203**, 249 (1999) [arXiv:hep-th/9804104].

[43] M. Mariño and G. W. Moore, "Counting higher genus curves in a Calabi-Yau manifold," Nucl. Phys. B **543**, 592 (1999) [arXiv:hep-th/9808131].

[44] D. Maulik and R. Pandharipande, "A topological view of Gromov-Witten theory," math.AG/0412503.

[45] D. Maulik and R. Pandharipande, "New calculations in Gromov–Witten theory," math.AG/0601395.

[46] P. Mayr and S. Stieberger, "Threshold corrections to gauge couplings in orbifold compactifications," Nucl. Phys. B **407**, 725 (1993) [arXiv:hep-th/9303017].

[47] G. W. Moore and E. Witten, "Integration over the u-plane in Donaldson theory," Adv. Theor. Math. Phys. **1**, 298 (1998) [arXiv:hep-th/9709193].

[48] J. F. Morales and M. Serone, "Higher derivative F-terms in N = 2 strings," Nucl. Phys. B **481**, 389 (1996) [arXiv:hep-th/9607193].

[49] V. Pestun and E. Witten, "The Hitchin functionals and the topological B-model at one loop," arXiv:hep-th/0503083.

[50] D. B. Ray and I. M. Singer, "Analytic Torsion for complex manifolds," Ann. of. Math. **98**, 154, (1973).

[51] N. Seiberg and E. Witten, "Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD," Nucl. Phys. B **431**, 484 (1994) [arXiv:hep-th/9408099].

[52] C. Vafa, "Instantons on D-branes," Nucl. Phys. B **463**, 435 (1996) [arXiv:hep-th/9512078].

[53] C. Voisin, "Miroirs et involutions sur les surfaces K3," Astérisque **218** (1993) 273.

[54] S. Yamaguchi and S. T. Yau, "Topological string partition functions as polynomials," JHEP **0407**, 047 (2004) [arXiv:hep-th/0406078].

[55] S. T. Yau and E. Zaslow, "BPS States, String Duality, and Nodal Curves on K3," Nucl. Phys. B **471**, 503 (1996) [arXiv:hep-th/9512121].